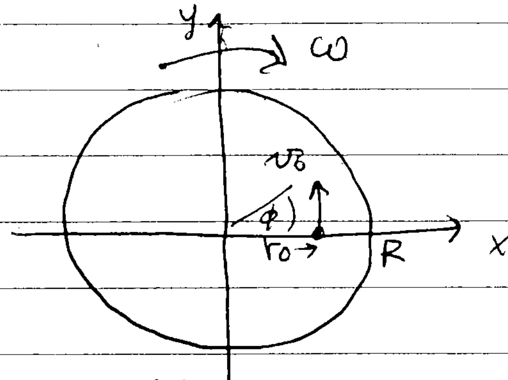
particle on frictionless rotating turn table



In the lab frame that is fixed total initial velocity is

$$\vec{v}_i = (v_0 - \omega r_0)\,\hat{y}$$

in the lab frame particle takes straight line trajectory

$$\Rightarrow \quad \vec{r}(t) = \vec{r}_0 + \vec{v}_i t$$

$$= r_0 \hat{x} + (v_0 - \omega r_0) t \hat{y}$$

The radial distance traveled at time t is

$$r(t) = \sqrt{x^2 + y^2} = \sqrt{r_0^2 + (v_0 - \omega r_0)^2 t^2}$$

$$= r_0 \sqrt{1 + \left(\frac{v_0}{\omega r_0} - 1\right)^2 \omega^2 t^2}$$

The angular rotation in the lab frame at time t is

$$\tan\phi_{fix} = \frac{y}{x} = \frac{(v_0 - \omega r_0)t}{r_0} = \left(\frac{v_0}{\omega r_0} - 1\right)\omega t$$

$$\phi_{fix} = \arctan\left(\left(\frac{v_0}{\omega r_0} - 1\right)\omega t\right)$$

In the rotating frame of reference

radial distance remains the same as above.
angular rotation becomes

$$\phi_{Rot} = \omega t + \phi_{fix}$$

we get trajectories as shown on following page

If $v_0 = 0$, ie just ~~drop~~ let go of particle

$$r(t) = \sqrt{r_0^2 + \omega^2 r_0^2 t^2} = r_0\sqrt{1+\omega^2 t^2}$$

$$\tan\phi_{fix} = -\omega t \quad \cancel{\phi_{fix} = -\tan^{-1}\omega t}$$

$$\phi_{rot} = \omega t \; - \arctan \omega t$$

If $\omega t \ll 1$, $\arctan\omega t \approx \omega t$ and $\phi_{rot} = 0$

particle hits edge when

$$R = r_0\sqrt{1+\omega^2 t^2}$$

$$\omega^2 t^2 = \frac{R^2}{r_0^2} - 1$$

when $\omega t = \sqrt{\frac{R^2}{r_0^2} - 1}$

So if r_0 is close to R, then $\omega t \ll 1$ and $\phi_{rot} \approx 0$

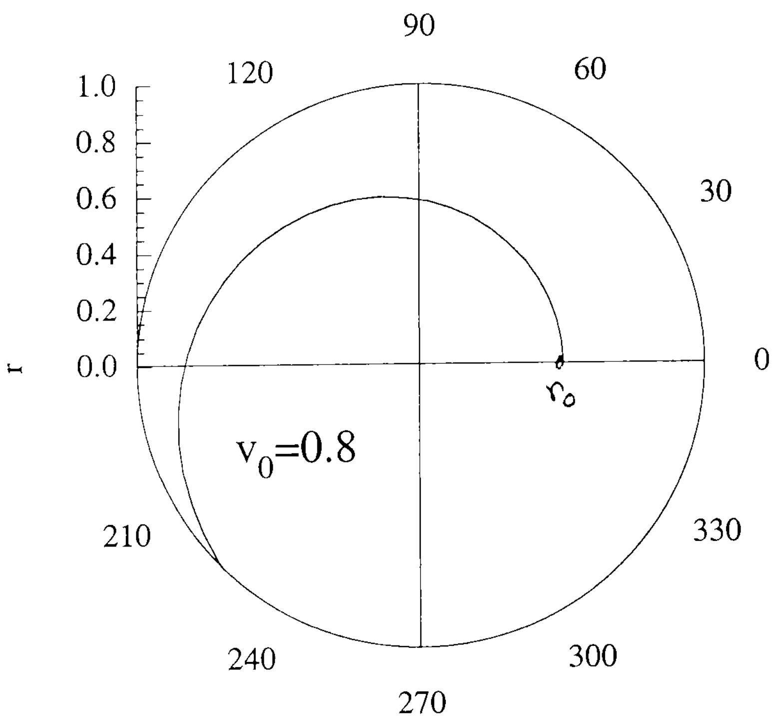
observer in rotating frame will not see he is rotating unless

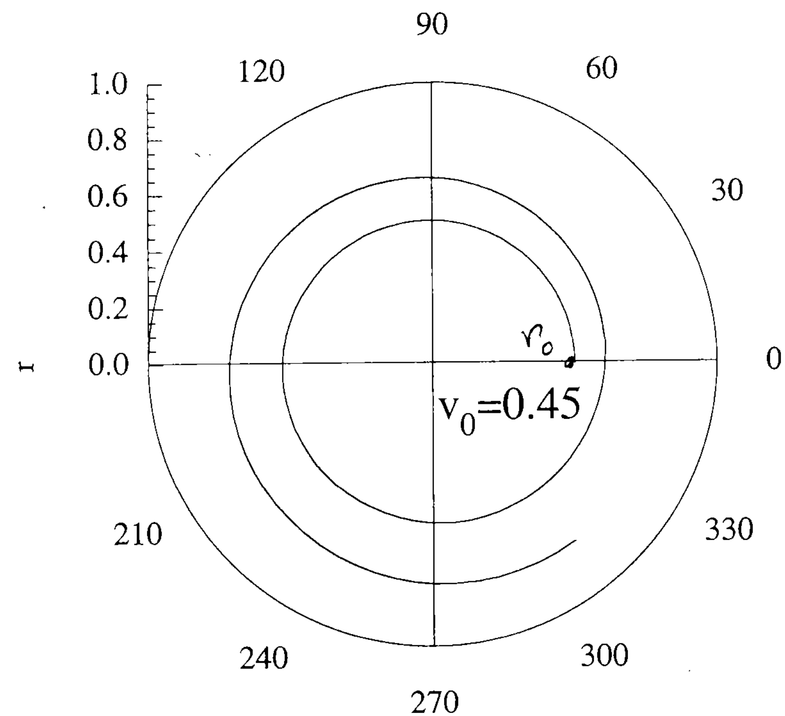
In general $\arctan(\omega t) < \omega t$ so $\phi_{rot} > 0$. Particle drifts counter clockwise as it falls trajectories as seen in the rotating frame.

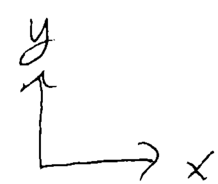
turn table rotates clockwise $\omega = 1$ rad/sec.

initial position is at $r_0 = \frac{1}{2}$

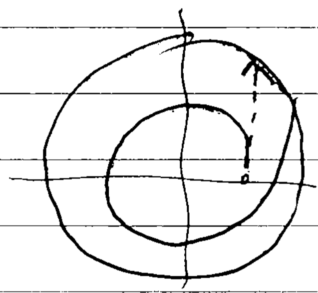
initial velocity is in $+y$ direction







To an observer at rest in the rotating frame, who is not aware that he is rotating, this behavior is extremely odd



If he throws a baseball straight, ~~aiming~~ aiming to hit the wall as in the dashed line in figure to the left, the ball instead wends around him following the solid curve

To explain such a trajectory, the observer would ~~to~~ conclude that the ball experienced a net force

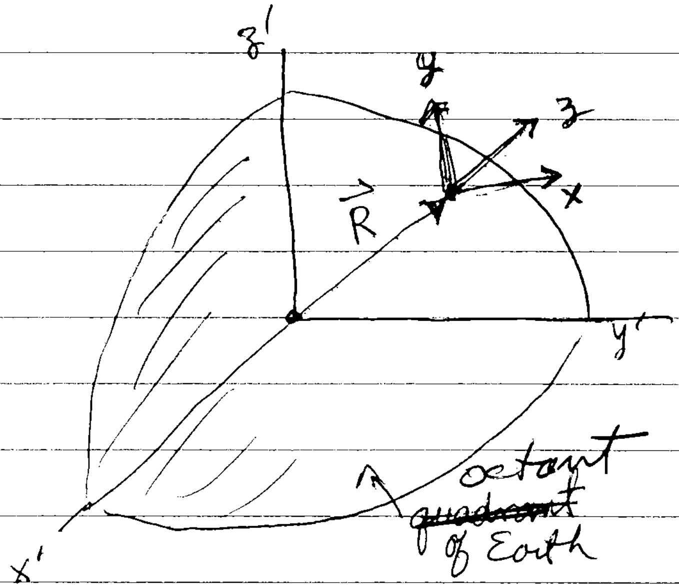
$$\vec{F} = m\omega^2 r\,\hat{r} - 2m\vec{\omega}\times\vec{v}_{rot}$$

centrifugal force — coriolis force

It is the Coriolis force that makes the ball curve around the observer.

Motion near the surface of the Earth

The motion of Earth about its axis of rotation means that the surface of the Earth is <u>not</u> an inertial frame of reference. Hence there will be effects due to centrifugal and Coriolis forces in predicting the trajectories of moving objects.



Consider as an inertial frame the coord system $x'y'z'$ with origin at center of Earth, and z' along axis of rotation (Ignore here the orbit of Earth around sun, which makes even this in principle a non-inertial frame.)

The rotating frame we take as the coord system xyz with origin fixed at a point on the surface of the Earth, and rotating with the Earth. $\hat{z}$ points normal to the surface of the Earth and $\hat{x}$ and $\hat{y}$ lie tangent to the surface.

$\vec{R}$ is the vector locating the origin of the rotating ~~system~~ frame with respect to the inertial frame.

Let $\vec{r}$ be the vector locating the position of an object with respect to the rotating frame.

The effective force $\vec{F}_{eff}$ determining the acceleration $\vec{a}_{rot}$ of an object, as measured in the rotating frame, is given by:

$$\vec{F}_{eff} = \vec{F} - m\ddot{\vec{R}}_{fix} - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}_{r}$$

where $\vec{\omega}$ is the angular velocity of the Earth rotating about its axis, $\vec{\omega} = \omega\hat{z}'$, $\omega = 7.3\times10^{-5}$ rad/s.

and $\vec{F} = \vec{F}_x + m\vec{g}_0$ where $\vec{g}_0 = -g\,\hat{e}_r'$, points radially inward, is the force of gravity at the surface

and $\vec{F}_x$ are any other non-gravitational forces acting on the object (these are the real, not the fictitious forces)

now:

$$\dot{\vec{R}}_{fix} = \left(\frac{d\vec{R}}{dt}\right)_{fix} = \vec{\omega}\times\vec{R}$$

and

$$\ddot{\vec{R}}_{fix} = \left(\frac{d\dot{\vec{R}}_{fix}}{dt}\right)_{fix} = \omega\times\dot{\vec{R}}_{fix} = \vec{\omega}\times(\vec{\omega}\times\vec{R})$$

So

$$\vec{F}_{eff} = \vec{F}_x + m\vec{g}_0 - m\vec{\omega}\times(\vec{\omega}\times(\vec{r}+\vec{R})) - m\dot{\vec{\omega}}\times\vec{r} - 2m\vec{\omega}\times\vec{v}_{r}$$

For Earth's rotation, we have $\dot{\vec{\omega}} \simeq 0$.

Define effective gravitational force as

$$\vec{g}_{eff} = \vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times(\vec{r}+\vec{R}))$$

non-gravitational forces

So $$\vec{F}_{eff} = \vec{F}_x + m\vec{g}_{eff} - 2m\,\vec{\omega}\times\vec{v}_{rot}$$

effective gravity — Coriolis force

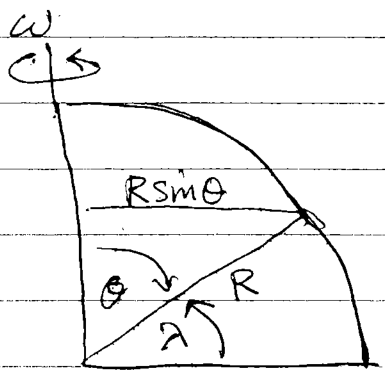
Effective Gravity

For objects near the surface of the Earth,

$$\vec{g}_{eff} = \vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times(\vec{r}+\vec{R}))$$
$$\simeq \vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times\vec{R})$$

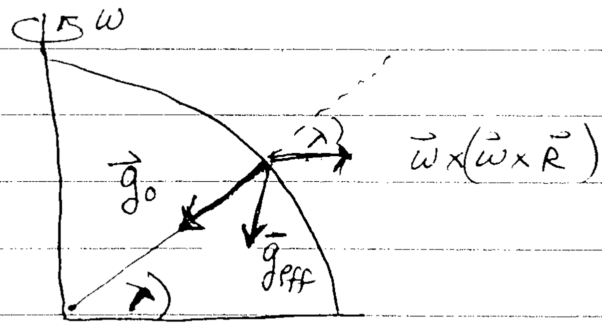
(in fact, this is the approx that leads from Newtons $\vec{F}_g = G\frac{m_1 m_2}{r^2}\hat{e}_r$ to a constant $\vec{g}$ near the surface of the Earth in the first place)

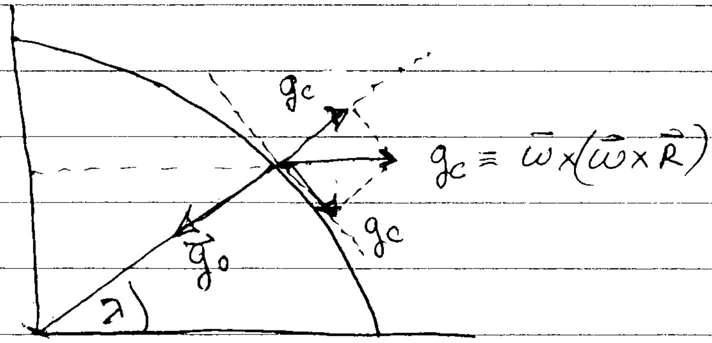
Note $\vec{g}_{eff}$ does not point radially inwards towards the center of the Earth, but rather varies in magnitude and direction with latitude.



$$|\vec{\omega}\times(\vec{\omega}\times\vec{R})| = \omega^2 R\sin\theta$$
$$= \omega^2 R\cos\lambda$$

where λ is the angle of latitude (Equator by definition is at $0°$)





Call the centrifugal contribution to $\vec{g}_{eff}$
$\vec{g}_c \equiv \vec{\omega}\times(\vec{\omega}\times\vec{R})$.

We can decompose $\vec{g}_c$ into its normal g_{cn}, and tangential, g_{ct}, components

$$g_{cn} = g_c \cos\lambda = \omega^2 R \cos^2\lambda$$
$$g_{ct} = g_c \sin\lambda = \omega^2 R \sin\lambda\cos\lambda = \tfrac{1}{2}\omega^2 R \sin(2\lambda)$$

Thus we have

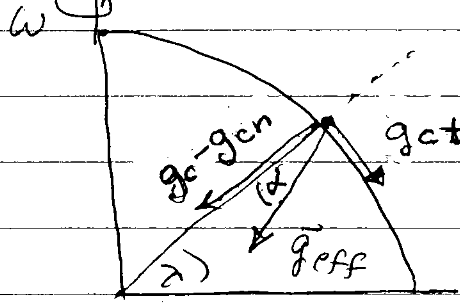
$$\vec{g}_{eff} = -(g_0 - g_{cn})\,\hat{e}_{r'} + g_{ct}\,\hat{e}_{\theta'}$$

$$= -(g_0 - \omega^2 R\cos^2\lambda)\,\hat{e}_{r'} + \omega^2 R\sin\lambda\cos\lambda\,\hat{e}_{\theta'}$$

Where $\hat{e}_{r'}$ and $\hat{e}_{\theta'}$ are the usual spherical coord basis vectors corresponding to the $x'y'z'$ vertical plane.

$$|\vec{g}_{eff}| = \left[(g_0 - \omega^2 R^2 \cos\lambda)^2 + (\omega^2 R\sin\lambda\cos\lambda)^2\right]^{1/2}$$

The angle by which $\vec{g}_{eff}$ deviates from the normal may be computed as



$$\tan\alpha = \frac{\vec{g}_{eff}\cdot\hat{e}_{\theta'}}{-\vec{g}_{eff}\cdot\hat{e}_{r'}} = \frac{\omega^2 R\sin\lambda\cos\lambda}{g_0 - \omega^2 R\cos^2\lambda}$$

$$\tan\alpha \simeq \alpha = \frac{\omega^2 R\cos\lambda\sin\lambda}{g_0 - \omega^2 R\cos^2\lambda}$$

$\alpha = 0$ at $\lambda = 0$ the pole, and $\lambda = \frac{\pi}{2}$ the equator

α is max at $\lambda = \frac{\pi}{4} = 45^\circ$

using values for R, g_0, ω, one finds

$\alpha(45^\circ) = 0.6'$ where $60' = 1^\circ$

problem 10-7

At the pole, $\lambda = \frac{\pi}{2}$, $\vec{g}_{eff} = -g_0\,\hat{e}_{r'}$

At the equator $\lambda = 0$; $\vec{g}_{eff} = -(g_0 - \omega^2 R)\,\hat{e}_{r'}$

$$|\vec{g}^{\,pole}_{eff}| - |\vec{g}^{\,equator}_{eff}| = \omega^2 R = 34\ mm/s^2$$

$$\omega = 7.3\times10^{-5}\ rad/s$$

$$R = 6370\ km$$

The actual difference is $52\ mm/s^2$. The discrepancy with our prediction is due to fact that Earth is not truly spherical - it is oblate spheroid, R greater at equator

Coriolis Force

$(g_c \ll g_0)$

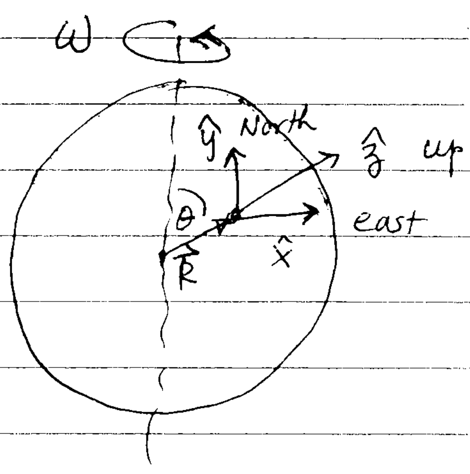
The effect of centrifugal force on $\vec{g}_{eff}$ is very small. If we neglect it, then the Coriolis force remains as the only "fictitious" force one needs to consider.

$$\vec{F}_{eff} = \vec{F}_x + m\vec{g} - 2m\,\vec{\omega} \times \vec{v}_{rot} = m\vec{a}_{rot}$$

For projectile motion, gravity is the only force that acts, hence $\vec{F}_x = 0$ and we have

$$\vec{a}_{rot} = \vec{g} - 2\vec{\omega} \times \vec{v}_{rot}$$

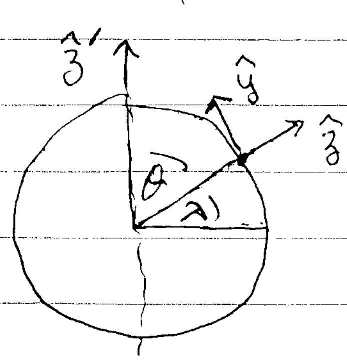
Choose the xyz coordinates at the Earth's surface as follows: $\hat{z}$ is radially outwards, ie "up", $\hat{x}$ points east, $\hat{y}$ points north



in terms of this basis, the angular velocity of rotation is

$$\vec{\omega} = \omega\hat{z}' = \omega\cos\theta\,\hat{z} + \omega\sin\theta\,\hat{y}$$

$$= \omega\sin\lambda\,\hat{z} + \omega\cos\lambda\,\hat{y},$$



where $\lambda = \frac{\pi}{2} - \theta$ is the angle of latitude

In terms of usual spherical coords
$\hat{z} = \hat{e}_{r'}$, $\hat{y} = -\hat{e}_{\theta'}$, $\hat{x} = \hat{e}_{\phi'}$

The Coriolis term in these coordinates is then

$$-2\vec{\omega}\times\vec{v}_{rot} = -2\,(\omega\cos\lambda\,\hat{y} + \omega\sin\lambda\,\hat{z})\times(\dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z})$$

↑ in rotating frame

$$= -2\left[-\dot{x}\,\omega\cos\lambda\,\hat{z} + \dot{z}\,\omega\cos\lambda\,\hat{x} + \dot{x}\,\omega\sin\lambda\,\hat{y} - \dot{y}\,\omega\sin\lambda\,\hat{x}\right]$$

$$= -2\omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda)\,\hat{x} - 2\omega\dot{x}\sin\lambda\,\hat{y} + 2\omega\dot{x}\cos\lambda\,\hat{z}$$

also, $\vec{g} = -g\,\hat{z}$

$$\vec{a}_{rot} = \vec{g} - 2\vec{\omega}\times\vec{v}_{rot}$$

$$\Rightarrow \quad (1) \quad \begin{cases} \ddot{x} = -2\omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) \\ \ddot{y} = -2\omega\dot{x}\sin\lambda \\ \ddot{z} = -g + 2\omega\dot{x}\cos\lambda \end{cases}$$

To solve these equations of motion, integrate to get

$$(2) \quad \begin{cases} \dot{x} = -2\omega(z\cos\lambda - y\sin\lambda) + v_{x0} \\ \dot{y} = -2\omega x\sin\lambda + v_{y0} \\ \dot{z} = -gt + 2\omega x\cos\lambda + v_{z0} \end{cases}$$

where v_{x0}, v_{y0}, v_{z0} are the initial velocities

Substitute ② into ① to get

$$\ddot{x} = -2\omega\left([-gt + 2\omega x\cos\lambda + v_{z0}]\cos\lambda + [2\omega x\sin\lambda - v_{y0}]\sin\lambda \right)$$

$$\ddot{y} = -2\omega\left[-2\omega(z\cos\lambda - y\sin\lambda) + v_{x0}\right]\sin\lambda$$

$$\ddot{z} = -g + 2\omega\left[-2\omega(z\cos\lambda - y\sin\lambda) + v_{x0}\right]\cos\lambda$$

these are small

Ignore terms of order ω^2 to get

$$\ddot{x} = 2\omega g t\cos\lambda - 2\omega v_{z0}\cos\lambda + 2\omega v_{y0}\sin\lambda$$

$$\ddot{y} = -2\omega v_{x0}\sin\lambda$$

$$\ddot{z} = -g + 2\omega v_{x0}\cos\lambda$$

Integrate twice to get

$$\begin{cases} x(t) = \dfrac{\omega g\cos\lambda}{3}t^3 - \omega t^2(v_{z0}\cos\lambda - v_{y0}\sin\lambda) + v_{x0}t + x_0 \\ y(t) = -\omega v_{x0}\sin\lambda\, t^2 + v_{y0}t + y_0 \\ z(t) = -\frac{1}{2}gt^2 + \omega v_{x0}\cos\lambda\, t^2 + v_{z0}t + z_0 \end{cases}$$

where x_0, y_0, z_0 are initial positions

We can always take the origin of the rotating frame so that $x_0 = y_0 = 0$

Examples

(1) Free fall from rest - object does not drop straight down
Coriolis force gives a deflection

$v_{x0} = v_{y0} = v_{z0} = 0$

$z_0 = h$ height the object is dropped from

$$\Rightarrow \begin{cases} x(t) = \dfrac{\omega g \cos\lambda}{3} t^3 \\ y(t) = 0 \\ z(t) = -\frac{1}{2} g t^2 + h \end{cases}$$

The object hits the ground at time t given by

$$z = 0 = -\tfrac{1}{2} g t^2 + h \Rightarrow t = \sqrt{\frac{2h}{g}}$$

deflection of object when it hits ground is

$$x = \frac{\omega g \cos\lambda}{3} \left(\frac{2h}{g}\right)^{3/2} = \boxed{\frac{\omega \cos\lambda}{3} \sqrt{\frac{8h^3}{g}}}$$

distance the objects drifts to the east in the course of its fall.

② Projectile fired vertically

$v_{x0} = v_{y0} = 0$, $v_{z0} = v_0$, $z_0 = 0$

$$\begin{cases} x(t) = \dfrac{\omega g \cos\lambda}{3} t^3 - \omega t^2 v_0 \cos\lambda \\ y(t) = 0 \\ z(t) = -\frac{1}{2} g t^2 + v_0 t \end{cases}$$

What is the net deflection of the projectile when it returns to hit the ground?

object will hit ground at t given by

$$z = 0 = -\tfrac{1}{2} g t^2 + v_0 t \Rightarrow t = \frac{2v_0}{g}$$

substitute this value of t into $x(t)$

$$\text{deflection } X = \omega \frac{\cos\lambda}{3} \left(\frac{2v_0}{g}\right)^3 - \omega\left(\frac{2v_0}{g}\right)^2 v_0 \cos\lambda$$

$$= \left(\frac{8}{3} \frac{v_0^3}{g^2} \cos\lambda - 4\frac{v_0^3}{g^2} \cos\lambda\right)\omega$$

$$X = -\frac{4}{3} \frac{v_0^3}{g^2} \omega \cos\lambda$$

to make this look more like the previous result, we note that the maximum height h occurs at time $t' = \frac{v_0}{g}$, so

$$h \equiv z_{max} = -\tfrac{1}{2} g \left(\frac{v_0^2}{g^2}\right) + v_0 \left(\frac{v_0}{g}\right) = \frac{v_0^2}{2g}$$

So $v_0 = \sqrt{2gh}$

deflection $x = -\frac{4}{3}\frac{\sqrt{8g^3h^3}}{g^2}\,\omega\cos\lambda$

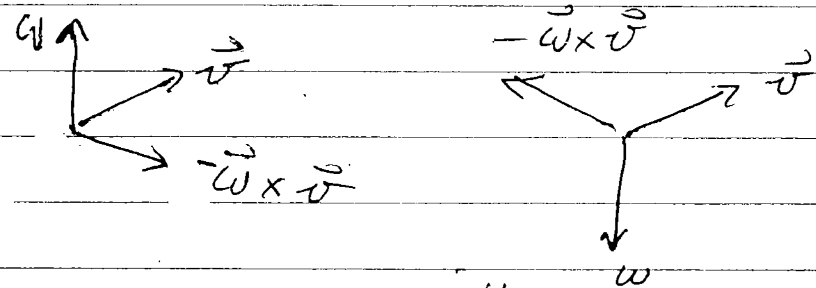
$$\boxed{x = -\frac{4}{3}\,\omega\cos\lambda\sqrt{\frac{8h^3}{g}}}$$ ← projectile drifts to the <u>west</u>

net deflection of projectile fired vertically is four time greater, and in <u>opposite</u> direction, to the deflection of a particle dropped from the same maximum height

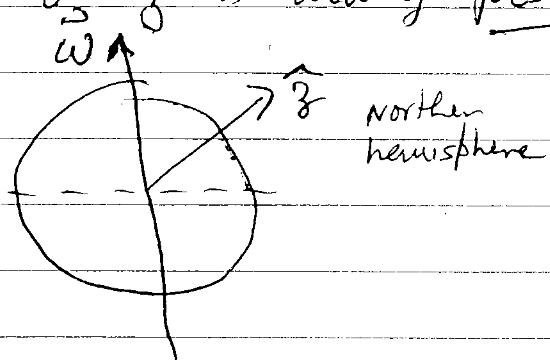
③ <u>Horizontal motion</u>

For motion in the horizonal x-y plane, we are only interested in the projection of $\vec{\omega}$ on the vertical axis, ie $\vec{\omega}\cdot\hat{z}$. This is because the component of $\vec{\omega}$ in the horizontal plane gives a Coriolis force $-\vec{\omega}\times\vec{v}_{rot}$ that is in the vertical direction, and so does not affect the horizontal motion

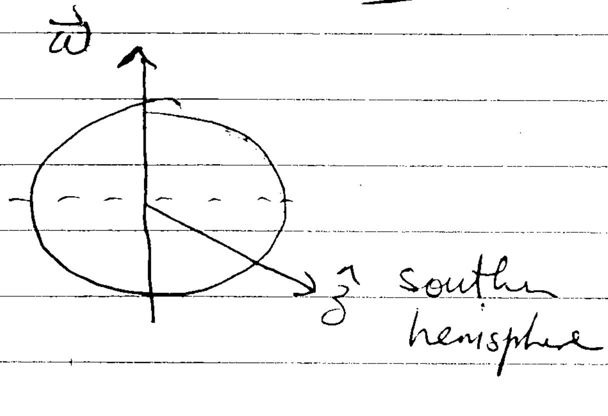
If there is an initial $\vec{v}$ in the x-y plane, the Coriolis force from the vertical component of $\vec{\omega}$ acts to the direction of $\vec{v}$. Since the Coriolis force is $-\vec{\omega}\times\vec{v}_{rot}$, if $\vec{\omega}\cdot\hat{z} > 0$, there will be a deflection in the clockwise direction. If $\vec{\omega}\cdot\hat{z} < 0$, the deflection is in the counter clockwise direction



In the Northern hemisphere, the projection of $\vec{\omega}$ on $\hat{z}$ is always positive – deflections are clockwise

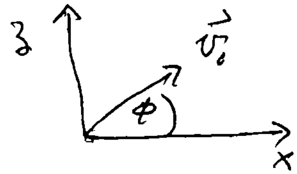


In the Southern hemisphere, the projection of $\vec{\omega}$ on $\hat{z}$ is always negative – deflections are counterclockwi



This effect is responsible for many important weather patterns due to air currents flowing & feeling the Coriolis force – see the text for details.

For the case of a projectile fired at angle ϕ in the easterly direction



$$v_{z0} = v_0 \sin\phi$$
$$v_{x0} = v_0 \cos\phi$$
$$v_{y0} = 0$$
$$z_0 = 0$$

$$x(t) = \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 v_{z0} \cos\lambda + v_{x0} t$$
$$y(t) = -\omega v_{x0} t^2 \sin\lambda$$
$$z(t) = -\frac{1}{2} g t^2 + v_{z0} t + \omega v_{x0} t^2 \cos\lambda$$

Because of the $y \propto t^2$, the trajectory does not remain in any given plane, but steadily drifts South words

projectile hits ground when

Coriolis force acts to change effective grav accel

$$z = 0 = -\left(\frac{1}{2} g - \omega v_{x0} \cos\lambda\right) t^2 + v_{z0} t = 0$$

$$\Rightarrow t_0 = \frac{2 v_{z0}}{g - 2\omega v_{x0} \cos\lambda}$$

can substitute in to find max height, $z(t_0/2)$, range, $x(t_0)$, and ~~s~~ sideways drift $y(t_0)$