Dynamics of Rigid Bodies

A rigid body is a collection of particles that maintain fixed displacements with respect to each other as the particles move.

To describe the motion of the rigid body we use two coordinate systems

1) an inertial frame of reference which stays fixed as the body moves
2) a rotating frame — now we call it the "body frame" that is fixed with respect to the body and rotates with respect to the inertial frame as the body rotates.

We can describe the motion of the body as follows

1) translation of a fixed point on the body — taken as the origin of the body frame — with respect to the inertial frame
2) orientation of the body frame with respect to the inertial frame.

This requires 6 coordinates — the three coordinates of the fixed pt on the body with respect to the inertial frame, and three angles to specify the orientation of the body with respect to the inertial frame.

let $\vec{r}$ be the position of a point on the body as measured in the body frame. In this body frame, the velocity of this point is

$$\vec{v}_{rot} = \left(\frac{d\vec{r}}{dt}\right)_{rot} = 0$$

since the body is stationary by definition in the body frame. The velocity of this point $\vec{v}$ in the inertial frame is then

$$\vec{v} = \vec{V} + \vec{v}_{rot} + \vec{\omega}\times\vec{r}$$

$$= \vec{V} + \vec{\omega}\times\vec{r}$$

$\vec{\omega}$ is instantaneous angular velocity of body.

where $\vec{V}$ is the velocity of the origin of the body frame with respect to the inertial frame.

Consider the kinetic energy of the solid body. The kinetic energy of particle α of the body is

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2 = \frac{1}{2} m_\alpha (\vec{V} + \vec{\omega}\times\vec{r}_\alpha)^2$$

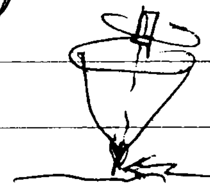
The total K.E. is then

$$T = \frac{1}{2}\sum_\alpha m_\alpha (\vec{V} + \vec{\omega}\times\vec{r}_\alpha)^2$$

$$= \frac{1}{2}\sum_\alpha \left[m_\alpha V^2 + 2 m_\alpha \vec{V}\cdot(\vec{\omega}\times\vec{r}_\alpha) + m_\alpha (\vec{\omega}\times\vec{r}_\alpha\right.$$

Consider the 2nd term. It will vanish if either

1) The origin of the body frame is chosen so that $\vec{V} = 0$. An example would be to choose the origin of the body frame of a spinning top to be the point where the top is in contact with the ground



origin of body frame

2) The origin of the body frame is chosen to be the center of mass. Then

$$\sum_\alpha m_\alpha \vec{V} \cdot (\vec{\omega} \times \vec{r}_\alpha) = \vec{V} \cdot \vec{\omega} \times \left(\sum_\alpha m_\alpha \vec{r}_\alpha \right)$$

center of mass

$$= \vec{V} \cdot \vec{\omega} \times (M \vec{R})$$

But $\vec{R} = 0$ if the origin is at the center of mass

One of the above two ~~[illegible]~~ choises ~~was~~ is usually ~~makes~~ made to simplify the description of the problem. In this case

$$T_{tran} \equiv \frac{1}{2} \sum_\alpha m_\alpha V^2 = \frac{1}{2} M V^2 \qquad \text{translational K.E.}$$

$$T_{rot} \equiv \frac{1}{2} \sum_\alpha m_\alpha (\vec{\omega} \times \vec{r}_\alpha)^2 \qquad \text{rotational K.E.}$$

$$T = T_{tran} + T_{rot}$$

Consider $T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha(\vec{\omega}\times\vec{r}_\alpha)^2$

once $(\vec{A}\times\vec{B})^2 = (\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) = A^2B^2 - (\vec{A}\cdot\vec{B})^2$

$$\Rightarrow T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha\left(\omega^2 r_\alpha^2 - (\vec{\omega}\cdot\vec{r}_\alpha)^2\right)$$

$$= \frac{1}{2}\sum_\alpha m_\alpha\left(\omega^2 r_\alpha^2 - (\vec{\omega}\cdot\vec{r}_\alpha)(\vec{r}_\alpha\cdot\vec{\omega})\right)$$

$$= \frac{1}{2}\vec{\omega}\cdot\underbrace{\sum_\alpha m_\alpha\left[r_\alpha^2\overleftrightarrow{\mathbb{1}} - \vec{r}_\alpha\vec{r}_\alpha\right]}_{\overleftrightarrow{I}}\cdot\vec{\omega}$$

2nd rank tensor

← the inertia tensor

$\overleftrightarrow{\mathbb{1}} \equiv$ identity tensor

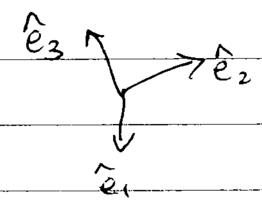
What is a tensor?

Review

a vector is an object that has magnitude and direction
we can represent the vector, without any reference to a particular coord system, in terms of an arrow.

$\vec{v}$ 

To do analytical calculations with vectors it is often easiest to choose a particular orthonormal set of basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and write $\vec{v}$ in terms of its projections on these basis vectors



$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$$

This is how we define the components of the vector $\vec{v}$, (v_1, v_2, v_3) or $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, in the coordinate system whose axes are along $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$v_i = \hat{e}_i\cdot\vec{v}$ gives component of $\vec{v}$ along ~~in the~~ the $\hat{e}_i$ axis

Tensor: Similarly one can think of a tensor as an object that exists independent of its representation in any particular coord system. A (2nd rank) tensor is a linear transformation that takes as input a vector $\vec{v}$, and gives as output another vector $\vec{w}$. For a tensor $\overleftrightarrow{I}$

$$\overleftrightarrow{I}\cdot\vec{v} = \vec{w}, \quad \text{linearity} \Rightarrow \overleftrightarrow{I}\cdot(a\vec{v}+b\vec{v}\,') = a\,\overleftrightarrow{I}\cdot\vec{v} + b\,\overleftrightarrow{I}\cdot\vec{v}\,'$$

To do analytical calculations, it is often easiest however to choose a particular coord system, with basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$, and write $\overleftrightarrow{I}$ in terms of its matrix representation in this coord system

$$I_{ij} \equiv \hat{e}_i\cdot(\overleftrightarrow{I}\cdot\hat{e}_j) = \hat{e}_i\cdot\overleftrightarrow{I}\cdot\hat{e}_j$$

Then $\vec{w} = \overleftrightarrow{I}\cdot\vec{v} = \overleftrightarrow{I}\cdot(v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3)$
$= v_1\overleftrightarrow{I}\cdot\hat{e}_1 + v_2\overleftrightarrow{I}\cdot\hat{e}_2 + v_3\overleftrightarrow{I}\cdot\hat{e}_3$

The ith component of the vector $\vec{w} = \overleftrightarrow{I}\cdot\vec{v}$ is

$$w_i = \hat{e}_i\cdot\overleftrightarrow{I}\cdot\vec{v} = v_1\,\hat{e}_i\cdot\overleftrightarrow{I}\cdot\hat{e}_1 + v_2\,\hat{e}_i\cdot\overleftrightarrow{I}\cdot\hat{e}_2 + v_3\,\hat{e}_i\cdot\overleftrightarrow{I}\cdot\hat{e}_3$$

$$w_i = \sum_{j=1}^{3} I_{ij}v_j$$

this is just matrix multiplication

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

We can also define what we mean by $\vec{w} \cdot \overleftrightarrow{I}$, ie dot product with the tensor from the left

If $\vec{v} \equiv \vec{w} \cdot \overleftrightarrow{I}$, then

$$\vec{v} \cdot \hat{e}_i = \vec{w}_i = (\vec{w} \cdot \overleftrightarrow{I}) \cdot \hat{e}_i = \vec{w} \cdot \overleftrightarrow{I} \cdot \hat{e}_i \qquad \vec{w} = \sum_{j=1}^{3} w_j \hat{e}_j$$

$$= \sum_{j=1}^{3} w_j \hat{e}_j \cdot \overleftrightarrow{I} \cdot \hat{e}_i$$

$$= \sum_{j=1}^{3} w_j I_{ji}$$

this is just matrix multiplication from the left

$$(w_1, w_2, w_3) \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} = (v_1, v_2, v_3)$$

The identity matrix $\overleftrightarrow{\mathbb{1}}$ has the property that $\overleftrightarrow{\mathbb{1}} \cdot \vec{v} = \vec{v}$ for any vector $\vec{v}$. So for <u>any</u> set of orthonormal basis vectors

$$\mathbb{1}_{ij} = \hat{e}_i \cdot \overleftrightarrow{\mathbb{1}} \cdot \hat{e}_j = \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

hence in any coordinate system the matrix of $\overleftrightarrow{\mathbb{1}}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix of the tensor $\vec{r}_\alpha \vec{r}_\alpha$ is given by

$$(\vec{r}_\alpha \vec{r}_\alpha)_{ij} = \hat{e}_i \cdot (\vec{r}_\alpha \ \vec{r}_\alpha) \cdot \hat{e}_j = (\hat{e}_i \cdot \vec{r}_\alpha)(\vec{r}_\alpha \cdot \hat{e}_j)$$

$= x_{\alpha i} x_{\alpha j}$ where $x_{\alpha i} \equiv \hat{e}_i \cdot \vec{r}_\alpha$ is component of $\vec{r}_\alpha$ in direction $\hat{e}_i$

If we call the coordinate basis directions $x\,y\,z$, ie $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$, $\hat{e}_3 = \hat{z}$ then the matrix of $\vec{r}_\alpha \vec{r}_\alpha$ is:

$$(\vec{r}_\alpha \vec{r}_\alpha) = \begin{pmatrix} x_\alpha x_\alpha & x_\alpha y_\alpha & x_\alpha z_\alpha \\ y_\alpha x_\alpha & y_\alpha y_\alpha & y_\alpha z_\alpha \\ z_\alpha x_\alpha & z_\alpha y_\alpha & z_\alpha z_\alpha \end{pmatrix} = \begin{pmatrix} x_\alpha^2 & x_\alpha y_\alpha & x_\alpha z_\alpha \\ y_\alpha x_\alpha & y_\alpha^2 & y_\alpha z_\alpha \\ z_\alpha x_\alpha & z_\alpha y_\alpha & z_\alpha^2 \end{pmatrix}$$

$$\vec{\omega} \cdot (\vec{r}_\alpha \vec{r}_\alpha) \cdot \vec{\omega} = (\omega_x, \omega_y, \omega_z) \begin{pmatrix} x_\alpha^2 & x_\alpha y_\alpha & x_\alpha z_\alpha \\ y_\alpha x_\alpha & y_\alpha^2 & y_\alpha z_\alpha \\ z_\alpha x_\alpha & z_\alpha y_\alpha & z_\alpha^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

If you do the matrix multiplications above you will see that

$$\vec{\omega} \cdot (\vec{r}_\alpha \vec{r}_\alpha) \cdot \vec{\omega} = (\vec{\omega} \cdot \vec{r}_\alpha)(\vec{r}_\alpha \cdot \vec{\omega}) = (\vec{\omega} \cdot \vec{r}_\alpha)^2$$

Inertia tensor $\overleftrightarrow{I}$

$$\overleftrightarrow{I} = \sum_\alpha m_\alpha \left[r_\alpha^2 \overleftrightarrow{\mathbb{1}} - \vec{r}_\alpha \vec{r}_\alpha \right]$$

in the $\hat{x}, \hat{y}, \hat{z}$ ~~basis~~ coord system this has the matrix representation

$$I_{ij} = \sum_\alpha m_\alpha \begin{pmatrix} r_\alpha^2 - x_\alpha^2 & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -y_\alpha x_\alpha & r_\alpha^2 - y_\alpha^2 & -y_\alpha z_\alpha \\ -z_\alpha x_\alpha & -z_\alpha y_\alpha & r_\alpha^2 - z_\alpha^2 \end{pmatrix}$$

since $r_\alpha^2 = x_\alpha^2 + y_\alpha^2 + z_\alpha^2$

we also can write

$$I_{ij} = \sum_\alpha m_\alpha \begin{pmatrix} y_\alpha^2 + z_\alpha^2 & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -y_\alpha x_\alpha & x_\alpha^2 + z_\alpha^2 & -y_\alpha z_\alpha \\ -z_\alpha x_\alpha & -z_\alpha y_\alpha & x_\alpha^2 + y_\alpha^2 \end{pmatrix}$$

For $i \neq j$ $\quad I_{ij} = -\sum_\alpha m_\alpha x_{\alpha i} x_{\alpha j}$

For $i = j$ $\quad I_{ii} = \sum_\alpha m_\alpha \left(r_\alpha^2 - x_{\alpha i}^2 \right)$

combining $\quad I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} r_\alpha^2 - x_{\alpha i} x_{\alpha j} \right)$

where $x_{\alpha 1} \equiv x_\alpha, \; x_{\alpha 2} \equiv y_\alpha, \; x_{\alpha 3} \equiv z_\alpha$

For a continuous mass distribution

$$I_{ij} = \int_V d^3r \, \rho(\vec{r}) \left(\delta_{ij} r^2 - x_i x_j \right)$$

$\overleftrightarrow{I}$ is similar to the electrostatic quadropole tensor of electrostatics, where in that case charge plays the role of mass

Just like a vector $\vec{v}$ represents three numbers $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ which gives the coordinates of the vector in some specified coordinate system, so

The tensor $\overleftrightarrow{I}$ represents the nine numbers $\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$ which gives the matrix representation of the tensor with respect to the specified coordinate system.

Note: $I_{ij} = I_{ji}$, inertia tensor is <u>symmetric</u> $\Rightarrow$ only 6 independent elements

If we know the elements I_{ij} of the tensor $\overleftrightarrow{I}$ with respect to a particular coordinate system, then one can apply a rotation transformation to this matrix to determine its elements with respect to a rotated coordinate system. We discuss the rotation transformation later.

Using the definition of the inertia tensor,

$$\boxed{\begin{aligned} T_{rot} &= \tfrac{1}{2}\,\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} \\ &= \tfrac{1}{2}\sum_{i,j=1}^{3} \omega_i\, I_{ij}\,\omega_j \end{aligned}}$$