If we choose our body coordinate frame so that $\vec{\omega}$ is aligned with a basis direction $\hat{e}_i$

then $\vec{\omega} = \omega \hat{e}_i$ and $\omega_j = \vec{\omega} \cdot \hat{e}_j = \omega \hat{e}_i \cdot \hat{e}_j = \omega \delta_{ij}$

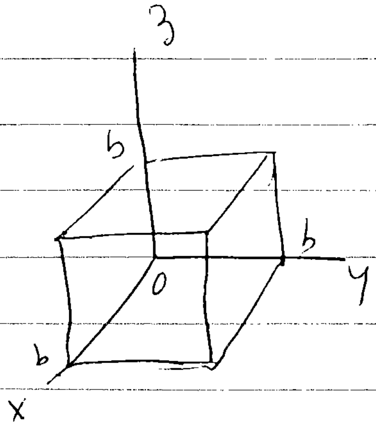
and $T_{rot} = \frac{1}{2} I_{ii} \omega^2$

We call $I_{ii} \equiv I$ the moment of inertia about the rotation axis $\hat{e}_i$. Then $T_{rot} = \frac{1}{2} I \omega^2$.

This is how you saw it in your earlier mechanics course where the moment of inertia was a scalar defined about a specific rotation axis.

Here we see that more generally the moment of inertia is a 3x3 tensor, which can be computed without reference to any particular rotation axis.

example inertia tensor of a homogeneous solid cube of mass density ρ, total mass M, length b, with respect to coordinates that have origin at a corner of the cube



$$I_{11} = \int_0^b dx \int_0^b dy \int_0^b dz \, (y^2 + z^2) \rho$$

$$= \rho b^2 \int_0^b dy \, y^2 + \rho b^2 \int_0^b dz \, z^2$$

$$= \frac{2\rho b^5}{3} = \frac{2}{3} M b^2 \quad \text{as } M = \rho b^3$$

Similarly $I_{22} = I_{33} = \frac{2}{3} M b^2$

$$I_{12} = -\int_0^b dx \int_0^b dy \int_0^b dz \; x y \rho$$

$$= -\rho b \left(\int_0^b dx\, x \right) \left(\int_0^b dy\, y \right)$$

$$= -\rho b \left(\frac{b^2}{2} \right) \left(\frac{b^2}{2} \right) = -\rho \frac{b^5}{4} = -\frac{1}{4} M b^2$$

Similarly $I_{13} = I_{23} = -\frac{1}{4} M b^2$

$$\overleftrightarrow{I} = M b^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$$

<u>Angular Momentum</u> <u>about origin of body frame</u>

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{V} + \vec{\omega} \times \vec{r}_\alpha)$$

$$= \left(\sum_\alpha m_\alpha \vec{r}_\alpha \right) \times \vec{V} + \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

$$= M \vec{R} \times \vec{V} + \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

if, as we did when we considered kinetic energy T, we take the origin of the body frame as either (1) a point at which $\vec{V} = 0$, or (2) the center of mass of the body so that $\vec{R} = 0$ in the body frame coordinates, then the first term will vanish.

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) \qquad \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$= \sum_\alpha m_\alpha \left[\vec{\omega}\,(\vec{r}_\alpha \cdot \vec{r}_\alpha) - \vec{r}_\alpha\,(\vec{r}_\alpha \cdot \vec{\omega}) \right]$$

$$= m_\alpha \left[r_\alpha^2 \overleftrightarrow{\mathbb{1}} - \vec{r}_\alpha \vec{r}_\alpha \right] \cdot \vec{\omega}$$

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \Rightarrow L_i = \sum_{j=1}^{3} I_{ij}\, \omega_j$$

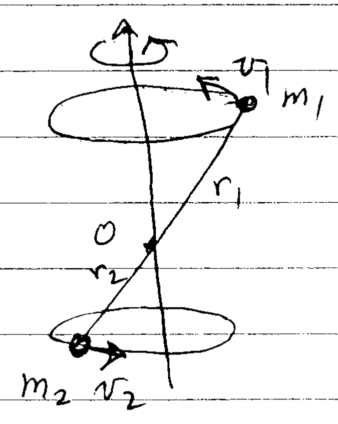
$$T_{rot} = \tfrac{1}{2}\, \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} = \tfrac{1}{2}\, \vec{\omega} \cdot \vec{L}$$

$$\boxed{\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}, \quad T_{rot} = \tfrac{1}{2}\, \vec{\omega} \cdot \vec{L}}$$

Note: Because $\overleftrightarrow{I}$ is a matrix, $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$

$\Rightarrow$ $\vec{L}$ is not necessarily in same direction as $\vec{\omega}$.

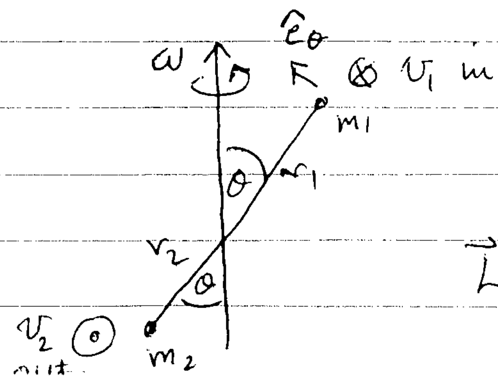
ex:



dumbell rotating

$$\vec{L} = m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2$$

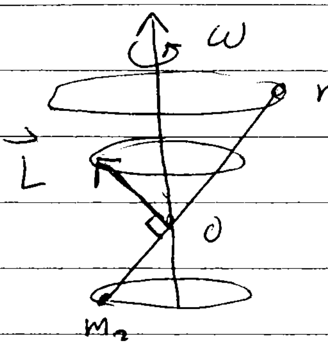
where $\vec{v}_1 = \vec{\omega} \times \vec{r}_1$, $\vec{v}_2 = \vec{\omega} \times \vec{r}_2$



$\vec{v}_1 = \omega r_1 \sin\theta$ into page

$\vec{v}_2 = \omega r_2 \sin\theta$ out of page

$$\vec{L} = (m_1 \omega r_1^2 \sin\theta + m_2 \omega r_2^2 \sin\theta)(-\hat{e}_\theta$$



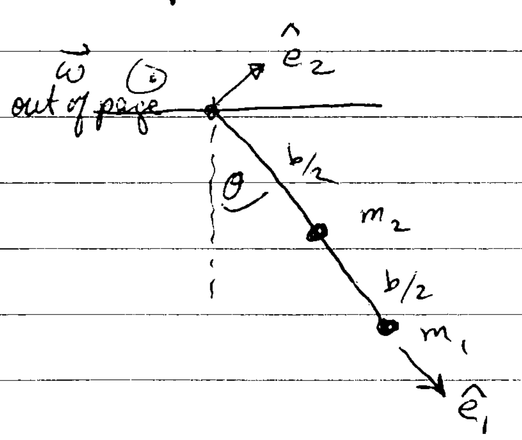
$\vec{L}$ rotates with angular freq ω and sweeps out a cone at angle $\frac{\pi}{2}-\theta$ with respect to $\vec{\omega}$.

We had from our discussion of dynamics of many particle systems, that if $\vec{L}$ is the total angular momentum

$$\frac{d\vec{L}}{dt} = \vec{N}_{ext}$$ where $\vec{N}_{ext}$ is the total external torque

In above dumbell example, since $\vec{L}$ is not const in time, (even though $\vec{\omega}$ is) there must be an external applied torque needed to keep the dumbell rotating.

<u>example</u>



$$\vec{\omega} = \omega\,\hat{e}_3 \qquad \hat{e}_3 \text{ is out of page}$$
$$= \dot{\theta}\,\hat{e}_3$$

Compute $\overleftrightarrow{I}$ in the $\hat{e}_1, \hat{e}_2, \hat{e}_3$ coordinate system.

Since the mass is all along the $\hat{e}_1$ axis, $y = z = 0 \rightarrow$ off diagonal terms of $\overleftrightarrow{I}$ all vanish.

call the $\hat{e}_1$ coord x
call the $\hat{e}_2$ coord y
call the $\hat{e}_3$ coord z

Also $I_{xx} = 0$ since this involves $y^2 + z^2$

$$I_{22} = \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) = \sum_\alpha m_\alpha (x_\alpha^2 + z_\alpha^2) = \sum_\alpha m_\alpha x_\alpha^2$$

$$= m_2 \left(\frac{b}{2}\right)^2 + m_1 b^2 = \left(m_1 + \frac{m_2}{4}\right) b^2$$

$$I_{33} = \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = \sum_\alpha m_\alpha x_\alpha^2$$

$$= \left(m_1 + \frac{m_2}{4}\right) b^2$$

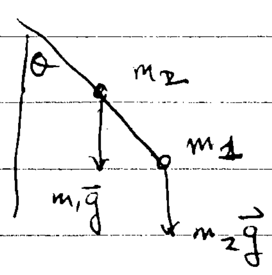
$$\Rightarrow \overleftrightarrow{I} = \left(m_1 + \frac{m_2}{4}\right) b^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} = \left(m_1 + \frac{m_2}{4}\right) b^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}$$

$$= \left(m_1 + \frac{m_2}{4}\right) b^2 \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}$$

$$\vec{L} = \left(m_1 + \frac{m_2}{4}\right) b^2 \dot{\theta}\, \hat{e}_3$$

$$\frac{d\vec{L}}{dt} = \vec{N}_{ext} = -\left(m_2 g \frac{b}{2} \sin\theta + m_1 g\, b \sin\theta\right) \hat{e}_3$$



torque $\vec{N} = \vec{r} \times \vec{F}$

$$\frac{d\vec{L}}{dt} = \vec{N}_{ext}$$

$$\Rightarrow \quad \left(m_1 + \frac{m_2}{4}\right) b^2 \ddot{\theta} = -\left(\frac{m_2}{2} + m_1\right) g b \sin\theta$$

$$\ddot{\theta} = -\left(\frac{m_1 + \frac{m_2}{2}}{m_1 + \frac{m_2}{4}}\right) \frac{g}{b} \sin\theta$$

this is just the equation of the ordinary pendulum with a slightly different prefactor. For small oscillations, $\theta \ll 1 \Rightarrow \sin\theta \simeq \theta$, we have single harmonic motion with angular frequency

$$\omega = \sqrt{\left(\frac{m_1 + \frac{m_2}{2}}{m_1 + \frac{m_2}{4}}\right) \frac{g}{b}}$$

We could also have gotten this from Lagrange's method

$$T = T_{rot} = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2}\left(m_1 + \frac{m_2}{4}\right) b^2 \dot{\theta}^2$$

$$U = -m_1 g b \cos\theta - m_2 g \frac{b}{2} \cos\theta$$

$$\mathcal{L} = T - U = \frac{1}{2}\left(m_1 + \frac{m_2}{4}\right) b^2 \dot{\theta}^2 + \left(m_1 + \frac{m_2}{2}\right) b g \cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \left(m_1 + \frac{m_2}{4}\right) b^2 \ddot{\theta}, \quad \frac{\partial \mathcal{L}}{\partial \theta} = -\left(m_1 + \frac{m_2}{2}\right) b g \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow \left(m_1 + \frac{m_2}{4}\right) b^2 \ddot{\theta} + \left(m_1 + \frac{m_2}{2}\right) b g \sin\theta = 0$$

gives the same equation of motion as above

Principle Axes of Rotation

The matrix representation of $\overleftrightarrow{I}$ in a given coord system with basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ is given by

$$I_{ij} = \hat{e}_i \cdot \overleftrightarrow{I} \cdot \hat{e}_j$$

in the examples we saw that calculations were easier if the matrix I_{ij} was diagonal.

Because $\overleftrightarrow{I}$ is symmetric, ie $I_{ij} = I_{ji}$, it is possible to show that one can always find a coord system with orthonormal basis vectors, $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ such that the matrix of $\overleftrightarrow{I}$ in this coord system is diagonal.

$$\hat{e}_i' \cdot \overleftrightarrow{I} \cdot \hat{e}_j' \equiv I_{ij}' = I_i \delta_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

diagonal matrix

The directions $\hat{e}_i'$ are called the principle axes of rotation. The diagonal values I_i are called the principle moments of inertia.

Suppose $\vec{\omega}$ is a vector parallel to a principle axis of rotation, ie $\vec{\omega} = \omega \hat{e}_j'$. Then

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} = \overleftrightarrow{I} \cdot \omega \hat{e}_j' = \omega \overleftrightarrow{I} \cdot \hat{e}_j'$$

The ith component of $\vec{L}$ in the coord system $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ is

$$L_i = \hat{e}_i' \cdot \vec{L} = \omega\, \hat{e}_i' \cdot \overleftrightarrow{I} \cdot \hat{e}_j' = \omega I_i \delta_{ij} = \begin{cases} 0 & i \neq j \\ \omega I_j & i = j \end{cases}$$

$$\vec{L} = \sum_{i=1}^{3} L_i \hat{e}_i' = \sum_i \omega I_i \delta_{ij} \hat{e}_i' = \omega I_j \hat{e}_j' = I_j \vec{\omega}$$

So $\vec{L} = I_j \vec{\omega}$

For an angular velocity $\vec{\omega}$ parallel to a principle axis of rotation, the angular momentum $\vec{L}$ is parallel to $\vec{\omega}$. The constant of proportionality is the corresponding moment of inertia

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \quad \Rightarrow \quad \boxed{\overleftrightarrow{I} \cdot \vec{\omega} = I_j \vec{\omega}} \quad \text{for } \vec{\omega} \text{ along } \hat{e}_j'$$

Finding the principle axes of ~~moment~~ inertia and their corresponding moments of inertia is the standard ~~eigenvalue~~ eigenvector / eigenvalue problem of linear algebra.

eigenvector / eigenvalue problem: Find a vector $\vec{v}$ such that

$$\overleftrightarrow{I} \cdot \vec{v} = \lambda \vec{v}$$

ie $\overleftrightarrow{I}$ operating on $\vec{v}$ gives back a vector parallel to $\vec{v}$ with proportionality constant λ. Then $\vec{v}$ is called an eigenvector and λ is the corresponding eigenvalue.

In this language, the principle axes of rotation are the eigenvectors of the inertia tensor $\overleftrightarrow{I}$ and the principle moments of inertia are the eigenvalues.

Before showing how to find the principle ~~eigenvectors and~~ axes and moments, we prove the following:

The eigenvectors of a symmetric tensor are orthogonal

proof: Suppose $\vec{v}_1$ is an eigenvector with eigenvalue λ_1
$\vec{v}_2$ is an eigenvector with eigenvalue λ_2

① if $\lambda_1 \neq \lambda_2$ then

$$\cancel{\vec{v}_1 \cdot \overleftrightarrow{I} \cdot \vec{v}_2 = \vec{v}_1 \cdot (\overleftrightarrow{I} \cdot \vec{v}_2) = \vec{v}_1 \cdot (\lambda_2 \vec{v}_2) = \lambda_2 \vec{v}_1 \cdot \vec{v}_2}$$

$$\vec{v}_1 \cdot \overleftrightarrow{I} \cdot \vec{v}_2 = \sum_{ij} v_{1i} I_{ij} v_{2j} = \sum_i v_{1i} \lambda_2 v_{2i} = \lambda_2 \vec{v}_1 \cdot \vec{v}_2$$

but since $I_{ij} = I_{ji}$ (since $\overleftrightarrow{I}$ is symmetric) we also have

$$\vec{v}_1 \cdot \overleftrightarrow{I} \cdot \vec{v}_2 = \sum_{ij} v_{1i} I_{ij} v_{2j} = \sum_{ij} v_{1i} I_{ji} v_{2j}$$

$$= \sum_{ij} v_{2j} I_{ji} v_{1i} = \sum_j v_{2j} \lambda_1 v_{1j} = \lambda_1 \vec{v}_1 \cdot \vec{v}_2$$

$\Rightarrow \lambda_2 \vec{v}_1 \cdot \vec{v}_2 = \lambda_1 \vec{v}_1 \cdot \vec{v}_2$

but if $\lambda_1 \neq \lambda_2$ this can only happen if $\vec{v}_1 \cdot \vec{v}_2 = 0$
ie $\vec{v}_1 \perp \vec{v}_2$

② if $\lambda_1 = \lambda_2$ then

any vector $\vec{w} = \alpha \vec{v}_1 + \beta \vec{v}_2$ is also an eigenvector with the same eigenvalue $\lambda_1 = \lambda_2$.

$\Rightarrow$ all vectors in the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors with the same eigenvalue. Therefore can always choose two orthogonal vectors in this plane to serve as basis vectors

This theorem tells us that the principle axes of inertia are always orthogo

Finding the eigenvectors + eigenvalues. If

$$\overleftrightarrow{I}\cdot\vec{\omega} = I\,\vec{\omega} \qquad \text{eigenvector } \vec{\omega} \text{ with eigenvalue } I$$

then $\left(\overleftrightarrow{I} - I\overleftrightarrow{\mathbb{1}}\right)\cdot\vec{\omega} = 0$ or $\begin{pmatrix} I_{11}-I & I_{12} & I_{13} \\ I_{21} & I_{22}-I & I_{23} \\ I_{31} & I_{32} & I_{33}-I \end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} =$

from linear algebra it can be shown that ~~if~~ the above can only be true for $\omega \neq 0$ if the determinant of the matrix is zero

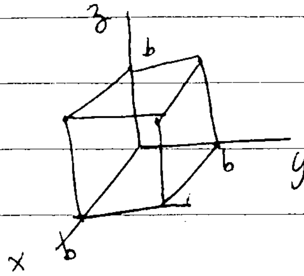
$$\begin{vmatrix} I_{11}-I & I_{12} & I_{13} \\ I_{21} & I_{22}-I & I_{23} \\ I_{31} & I_{32} & I_{33}-I \end{vmatrix} = 0$$

evaluating this determinant gives a cubic equation in I known as the "characteristic equation". The roots of this cubic equation are the three eigenvalues, ie the principle moments of inertia I_1, I_2, I_3. Having found a given eigenvalue I_i, then one substitutes this into

$$\begin{pmatrix} I_{11}-I_i & I_{12} & I_{13} \\ I_{21} & I_{22}-I_i & I_{23} \\ I_{31} & I_{32} & I_{33}-I_i \end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = 0$$

to solve for the components ω_i of the corresponding eigenvectors

In this way one finds the principle axes of inertia principle axes of a solid cube



$$I_{ij} = Mb^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$$

Could find eigenvalues via characteristic equation – see text but for matrices that are very symmetric, like the above, it is sometimes easier just to guess.

Try $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ then $\overleftrightarrow{I} \cdot \vec{v}_1 = Mb^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\overleftrightarrow{I} \cdot \vec{v}_1 = Mb^2 \begin{pmatrix} \frac{2}{3} - \frac{2}{4} \\ \frac{2}{3} - \frac{2}{4} \\ \frac{2}{3} - \frac{2}{4} \end{pmatrix} = \frac{Mb^2}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \frac{2}{3} - \frac{2}{4} = \frac{8-6}{12} = \frac{2}{12} = \frac{1}{6}$$

So $\vec{v}_1$ is eigenvector with eigenvalue $I_1 = \frac{Mb^2}{6}$.

One principle axis is $\hat{e}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The next principle axis is orthogonal to the above one.

Try $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\overleftrightarrow{I} \cdot \vec{v}_2 = Mb^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = Mb^2 \begin{pmatrix} \frac{2}{3} + \frac{1}{4} \\ -\frac{1}{4} + \frac{1}{4} \\ -\frac{1}{4} - \frac{2}{3} \end{pmatrix} = \frac{11}{12} Mb^2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\frac{2}{3} + \frac{1}{4} = \frac{8+3}{12} = \frac{11}{12}$$

So $\vec{v}_2$ is eigenvector with eigenvalue $I_2 = \frac{11}{12} Mb^2$

~~finally~~ $\hat{e}_2 = \dfrac{\vec{v}_2}{|\vec{v}_2|} = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$

finally, the last principle axis should be orthogonal to both $\vec{v}_1$ and $\vec{v}_2$ above. $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\end{pmatrix}$

$$\overleftrightarrow{I}\cdot\vec{v}_3 = Mb^2\begin{pmatrix}\frac{2}{3} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{2}{3} & -1/4\\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}-1\\2\\-1\end{pmatrix} = Mb^2\begin{pmatrix}-\frac{2}{3}-\frac{2}{4}+\frac{1}{4}\\ +\frac{1}{4}+\frac{4}{3}+\frac{1}{4}\\ \frac{1}{4}-\frac{2}{4}-\frac{2}{3}\end{pmatrix}$$

$$= \frac{Mb^2}{12}\begin{pmatrix}-8-6+3\\ 3+16+3\\ 3-6-8\end{pmatrix} = \frac{Mb^2}{12}\begin{pmatrix}-11\\22\\-11\end{pmatrix} = \frac{11}{12}Mb^2\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

so $\vec{v}_3$ is eigenvector with eigenvalue $I_3 = I_2 = \frac{11}{12}Mb^2$

$$\hat{e}_3' = \frac{\vec{v}_3}{|\vec{v}_3|} = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

in the basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$ $\overleftrightarrow{I}$ has the matrix

$$I_{ij} = \frac{M}{12}b^2\begin{pmatrix}2 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{pmatrix}$$