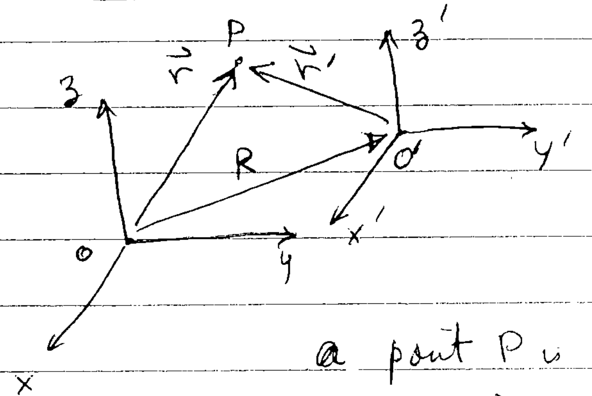
If we have computed $\overleftrightarrow{I}$ in one coord system, how can we find its values in another coord system

① Parallel Axis Theorem - translation of coord axes




consider two coordinate systems, $xyz$ and $x'y'z'$, which have the same orientation of basis vectors, but are separated by a displacement $\vec{R}$.

a point P is given by the ~~vector~~ point vector $\vec{r}$ in the $xyz$ system, and by the vector $\vec{r}' = \vec{r} - \vec{R}$ in the $x'y'z'$ coord system.

in the $xyz$ system

$$I_{ij} = \sum_\alpha m_\alpha \left[ r_\alpha^2 \delta_{ij} - x_{\alpha i} x_{\alpha j} \right]$$

in the $x'y'z'$ system

$$I'_{ij} = \sum_\alpha m_\alpha \left[ r_\alpha'^2 \delta_{ij} - x'_{\alpha i} x'_{\alpha j} \right]$$

substitute $r_\alpha'^2 = (\vec{r}_\alpha - \vec{R})^2 = r_\alpha^2 + R^2 - 2\vec{r}_\alpha \cdot \vec{R}$

$$x'_{\alpha i} x'_{\alpha j} = (x_{\alpha i} - R_i)(x_{\alpha j} - R_j) = x_{\alpha i} x_{\alpha j} + R_i R_j - x_{\alpha i} R_j - x_{\alpha j} R_i$$

$$\Rightarrow I'_{ij} = \sum_\alpha m_\alpha \left[ \delta_{ij} r_\alpha^2 - x_{\alpha i} x_{\alpha j} \right]$$

$$+ \sum_\alpha m_\alpha \left[ \delta_{ij} (R^2 - 2\vec{r}_\alpha \cdot \vec{R}) - R_i R_j + x_{\alpha i} R_j + x_{\alpha j} R_i \right]$$

$$= I_{ij} + \delta_{ij} \left[ M R^2 - 2\left(\sum_\alpha m_\alpha \vec{r}_\alpha\right) \cdot \vec{R} \right] - M R_i R_j$$

$$+ \left(\sum_\alpha m_\alpha x_{\alpha i}\right) R_j + \left(\sum_\alpha m_\alpha x_{\alpha j}\right) R_i$$

If we choose $x'y'z'$ to be the center of mass coordinate system, ie the origin $O'$ is at the center of mass of the body, then $\vec{R}$ is the center of mass position in the inertial fixed frame, and $\sum_\alpha m_\alpha x_{\alpha i} = MR_i$

then $I'_{ij} = I_{ij} + M(\delta_{ij} R^2 - R_i R_j) - 2MR^2 \delta_{ij} + MR_i R_j + MR_j R_i$

$= I_{ij} + M(\delta_{ij} R^2 - R_i R_j) - 2M(\delta_{ij} R^2 - R_i R_j)$

$$\boxed{\underbrace{I'_{ij}}_{\text{in CM frame}} = I_{ij} - M(\delta_{ij} R^2 - R_i R_j)}$$

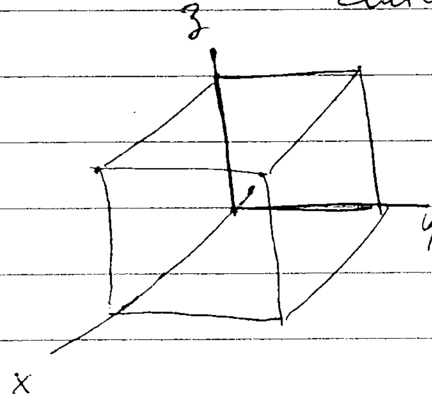
parallel axis theorem relates $\overleftrightarrow{I}$ in the center of mass frame to $\overleftrightarrow{I}$ in a displaced frame

or equivalently

$$I_{ij} = \underbrace{I'_{ij}}_{\text{in CM frame}} + M(\delta_{ij} R^2 - R_i R_j)$$

where $\vec{R}$ is location of origin of primed coord system as measured in unprimed coord syst
$R$ is also location of CM

example: we computed $I_{ij}$ for a solid cube with coords centered on one corner. What is $I'_{ij}$ for coords centered at the center of the cube?




cube's center is at $\vec{R} = \frac{b}{2}(\hat{x} + \hat{y} + \hat{z})$ with respect to the unprimed coord system

$R^2 = \frac{b^2}{4} 3$

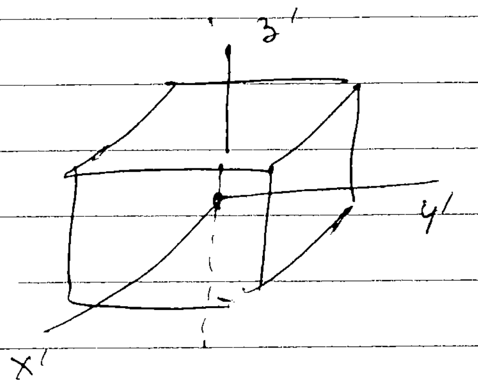
$I'_{ij} = I_{ij} - M(\delta_{ij} R^2 - R_i R_j) =$

$$= Mb^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} - Mb^2 \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$I'_{ij} = Mb^2 \begin{pmatrix} \frac{2}{3}-\frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3}-\frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3}-\frac{1}{2} \end{pmatrix}$$

$$= Mb^2 \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} = \frac{Mb^2}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$I'_{ij}$ is diagonal in the primed coord system




With respect to the origin at the center of the cube, the inertia tensor is

$$\overleftrightarrow{I}' = \frac{Mb^2}{6} \overleftrightarrow{\mathbb{1}}$$ prop to the identity tensor!

all principle moments of inertia (eigenvalues) are equal. $\Rightarrow \overleftrightarrow{I}'$ is diagonal in <u>any</u> coord system whose origin is at the center of the cube.

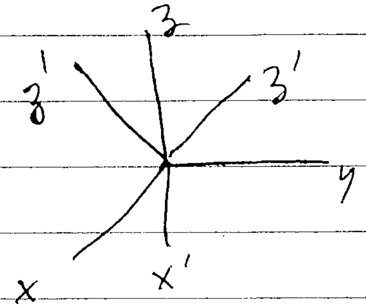
angular momentum about center of cube will be

$$\vec{L}' = \overleftrightarrow{I} \cdot \vec{\omega} = \frac{Mb^2}{6} \vec{\omega}$$

$\vec{L}'$ always parallel to $\vec{\omega}$ no matter what is the direction of $\vec{\omega}$.

<u>Moral:</u> The principle moments of inertia depend on where you choose to put the origin of the coord system

② rotation of coord axes




If know $I_{ij}$ in the $xyz$ system, what is $I'_{ij}$ in the $x'y'z'$ system?

$$I'_{ij} = \hat{e}'_i \cdot \overleftrightarrow{I} \cdot \hat{e}'_j$$

we can always expand the $\hat{e}'_i$ in the basis vectors $\hat{e}_j$

$$\hat{e}'_i = \sum_l (\hat{e}'_i \cdot \hat{e}_l)\, \hat{e}_l$$

$$\hat{e}'_j = \sum_k (\hat{e}'_j \cdot \hat{e}_k)\, \hat{e}_k$$

So $I'_{ij} = \hat{e}'_i \cdot \overleftrightarrow{I} \cdot \hat{e}'_j = \sum_{l,k} (\hat{e}'_i \cdot \hat{e}_l)\, \hat{e}_l \cdot \overleftrightarrow{I} \cdot \hat{e}_k\, (\hat{e}'_j \cdot \hat{e}_k)$

$$= \sum_{l,k} (\hat{e}'_i \cdot \hat{e}_l)\, I_{lk}\, (\hat{e}_k \cdot \hat{e}'_j)$$

Now recall that $(\hat{e}'_i \cdot \hat{e}_l) = \lambda_{il}$ is the rotation matrix from the unprimed system to the primed system.

$(\hat{e}_k \cdot \hat{e}'_j) = \lambda_{jk} = (\lambda^t)_{kj}$ $\quad\lambda^t$ is transpose of rotation matrix

$$I'_{ij} = \sum_{lk} \lambda_{il}\, I_{lk}\, \lambda^t_{kj} = \lambda \cdot I \cdot \lambda^t$$

matrix multiplication

the above rule is how tensors transform under rotation

Transformation under rotation

vector: $v_i' = \sum_j \lambda_{ij} v_j$

tensor $I_{ij}' = \sum_{lk} \lambda_{il} I_{lk} \lambda^t_{kj}$

or $= \sum_{lk} I_{lk} \lambda_{il} \lambda_{jk}$

## Euler's eqn for rotational motion

We want to find dynamic equations that describe how the solid body rotates when an external torque is applied. This will come from

$$\vec{N}_{ext} = \left(\frac{d\vec{L}}{dt}\right)_{fix}$$

$\vec{N}_{ext}$ is the total external torque applied to the body

$\left(\frac{d\vec{L}}{dt}\right)_{fix}$ is the rate of change of the angular momentum, as seen in the fixed inertial frame of reference.

(this eqn only holds in inertial frames)

We can now write

$$\left(\frac{d\vec{L}}{dt}\right)_{fix} = \left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega}\times\vec{L}$$

where $\left(\frac{d\vec{L}}{dt}\right)_{rot}$ is rate of change of angular momentum as seen in the body frame

$$\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega}$$

Since $\overleftrightarrow{I}$ is computed in the body frame, it is independent of time in the body frame

$$\left(\frac{d\vec{L}}{dt}\right)_{rot} = \left(\frac{d(\overleftrightarrow{I}\cdot\vec{\omega})}{dt}\right)_{rot} = \overleftrightarrow{I}\cdot\left(\frac{d\omega}{dt}\right)_{rot} = \overleftrightarrow{I}\cdot\dot{\vec{\omega}}$$

(remember $\left(\frac{d\vec{\omega}}{dt}\right)_{rot} = \left(\frac{d\vec{\omega}}{dt}\right)_{fix}$ } so we just write $\dot{\vec{\omega}}$ for either

$$\vec{\omega}\times\vec{L} = \vec{\omega}\times(\overleftrightarrow{I}\cdot\vec{\omega})$$

$$\left(\frac{d\vec{L}}{dt}\right)_{fix} = \overleftrightarrow{I}\cdot\dot{\vec{\omega}} + \vec{\omega}\times(\overleftrightarrow{I}\cdot\vec{\omega})$$

If we use as the ~~axes~~ coordinate axes of the body frame the principle axes of rotation, then

$$\overleftrightarrow{I}\cdot\dot{\vec{\omega}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{pmatrix}$$

$$\vec{\omega}\times(\overleftrightarrow{I}\cdot\vec{\omega}) = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} = \begin{pmatrix} I_3\omega_3\omega_2 - I_2\omega_2\omega_3 \\ I_1\omega_1\omega_3 - I_3\omega_3\omega_1 \\ I_2\omega_2\omega_1 - I_1\omega_1\omega_2 \end{pmatrix}$$

$$= \begin{pmatrix} (I_3 - I_2)\omega_3\omega_2 \\ (I_1 - I_3)\omega_1\omega_3 \\ (I_2 - I_1)\omega_2\omega_1 \end{pmatrix}$$

Combining the pieces with $\vec{N}^{ext} = \vec{\omega}\times(\overleftrightarrow{I}\cdot\vec{\omega}) + \overleftrightarrow{I}\cdot\dot{\vec{\omega}}$ we get

$$\begin{cases} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 = N_1^{ext} \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = N_2^{ext} \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = N_3^{ext} \end{cases}$$

When no external torque is applied, the above
<u>Euler's equations for force-free rotation</u>

$$\begin{cases} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_3\omega_2 = 0 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_2\omega_1 = 0 \end{cases}$$

## Stability of force-free solid body rotations

We want to look at the stability of force free rotations of a solid body about its principle axes of inertia

Suppose a body is rotating about one of the principle axes of inertia, for example $\hat{e}_1$,

$$\vec{\omega} = \omega_1 \hat{e}_1$$

We now ask what happens if $\vec{\omega}$ has a small perturbation in the $\hat{e}_2$ or $\hat{e}_3$ directions

$$\vec{\omega} = \omega_1 \hat{e}_1 + \delta\omega_2 \hat{e}_2 + \delta\omega_3 \hat{e}_3 \qquad \delta\omega_2 \ll \omega_1, \ \delta\omega_3 \ll \omega_1$$

Euler's equations become

1) $I_1 \dot{\omega}_1 + (I_3 - I_2)\,\delta\omega_2 \delta\omega_3 \approx I_1 \dot{\omega}_1 = 0$

2) $I_2 \delta\dot{\omega}_2 + (I_1 - I_3)\,\omega_1 \delta\omega_3 = 0$

3) $I_3 \delta\dot{\omega}_3 + (I_2 - I_1)\,\omega_1 \delta\omega_2 = 0$

In the first line we set $\delta\omega_2 \delta\omega_3 \approx 0$ since this is second order in the perturbation. Here we are doing a <u>linear stability analysis</u> ie keeping only the leading (linear) terms in the perturbation.

1) $\Rightarrow$ $\dot{\omega}_1 = 0$ or $\omega_1 = $ constant

2) $\Rightarrow$ $\delta\dot{\omega}_2 + a\,\delta\omega_3 = 0$ where $a = \left(\frac{I_1 - I_3}{I_2}\right)\omega_1$

3) $\Rightarrow$ $\delta\dot{\omega}_3 = b\,\delta\omega_2$ where $b = \left(\frac{I_1 - I_2}{I_3}\right)\omega_1$

2) $\Rightarrow$ $\delta\ddot{\omega}_2 + a\,\delta\dot{\omega}_3 = 0$

substitute from 3) $\Rightarrow$ $\delta\ddot{\omega}_2 + ab\,\delta\omega_2 = 0$

Similarly $\delta\ddot{\omega}_3 + ab\,\delta\omega_3 = 0$

The above look just like equation for simple harmonic oscillator, provided $ab$ is a positive constant. If $ab > 0$, then $\delta\omega_2$ and $\delta\omega_3$ will oscillate about zero with an angular frequency of

$$\Omega = \sqrt{ab} = \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}\;\omega_1$$

The rotation about the $\hat{e}_1$ axis will be stable.

If, however, $ab < 0$, then the solution to

$$\delta\ddot{\omega}_2 + ab\,\delta\omega_2 = 0$$

is of the form

$$\delta\omega_2(t) = A e^{-\lambda t} + B e^{\lambda t} \qquad \text{with } \lambda = \sqrt{|ab|}$$

and the second term will cause the perturbation to grow exponentially – the rotation about $\hat{e}_1$ axis will be unstable.

Now:

$$ab = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\,\omega_1^2$$

If $I_1$ is the largest of all the principle moments of inertia then $(I_1 - I_3)(I_1 - I_2) > 0$, and so $ab > 0$, rotation is stable

If $I_1$ is the smallest of all the principle moments of inertia, then $(I_1 - I_3) < 0$ and $(I_1 - I_2) < 0$, but $(I_1 - I_3)(I_1 - I_2) > 0$, so $ab > 0$, rotation is stable

If $I_1$ is the middle of the three values of the principle moments of inertia, then $(I_1 - I_3)(I_1 - I_2) < 0$, so $ab < 0$, rotation is unstable

Conclusion: rotation about principle axis corresponding to the largest and the smallest principle moments is always stable. Rotation about the principle axis corresponding to the middle moment is always unstable.