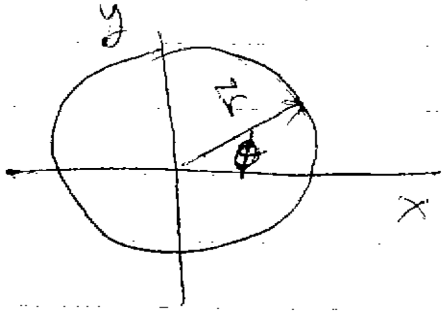
velocity + acceleration

particles trajectory $\vec{r}(t) = x_1(t)\hat{e}_1 + x_2(t)\hat{e}_2 + x_3(t)\hat{e}_3$

velocity $\vec{v}(t) = \dfrac{d\vec{r}}{dt}(t) \equiv \dot{\vec{r}}(t) = \dot{x}_1(t)\hat{e}_1 + \dot{x}_2(t)\hat{e}_2 + \dot{x}_3(t)\hat{e}_3$

acceleration $\vec{a}(t) = \dfrac{d^2\vec{r}}{dt^2}(t) \equiv \ddot{\vec{r}}(t) = \ddot{x}_1(t)\hat{e}_1 + \ddot{x}_2(t)\hat{e}_2 + \ddot{x}_3(t)\hat{e}_3$

example: a particle moving in uniform circular motion in the xy plane:



$\phi(t) = \omega t$ ω = angular speed

$\vec{r}(t) = r\cos(\omega t)\hat{e}_1 + r\sin(\omega t)\hat{e}_2$

$\Rightarrow$ $\vec{v} = -\omega r\sin(\omega t)\hat{e}_1 + \omega r\cos(\omega t)\hat{e}_2$

$\vec{a} = -\omega^2 r\cos(\omega t)\hat{e}_1 - \omega^2 r\sin(\omega t)\hat{e}_2 = -\omega^2\vec{r}(t)$

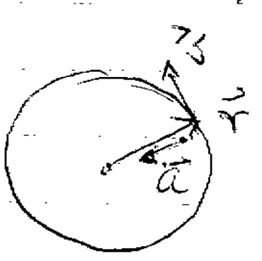
$\vec{a}$ inward radial

$|\vec{v}|^2 = \omega^2 r^2\sin^2(\omega t) + \omega^2 r^2\cos^2(\omega t) = \omega^2 r^2$

$v = \omega r$

$\vec{v}\cdot\vec{r} = -\omega r^2\cos(\omega t)\sin(\omega t) + \omega r^2\sin(\omega t)\cos(\omega t) = 0$

$\Rightarrow \vec{v} \perp \vec{r}$

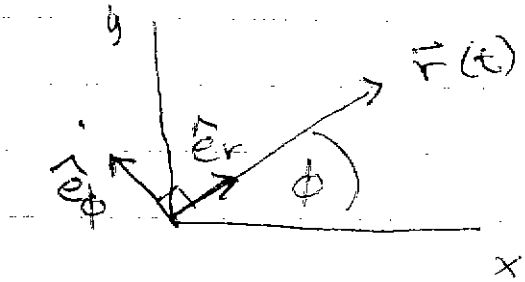


Combine $v = \omega r$ with $a = \omega^2 r \Rightarrow a = \dfrac{v^2}{r}$

familiar centripetal acceleration

2D polar coordinates

Consider particle with trajectory $\vec{r}(t)$.
At any t we can define a set of basis vectors $\hat{e}_r$ and $\hat{e}_\phi$ that point parallel to $\vec{r}$ and orthogonal to $\vec{r}$



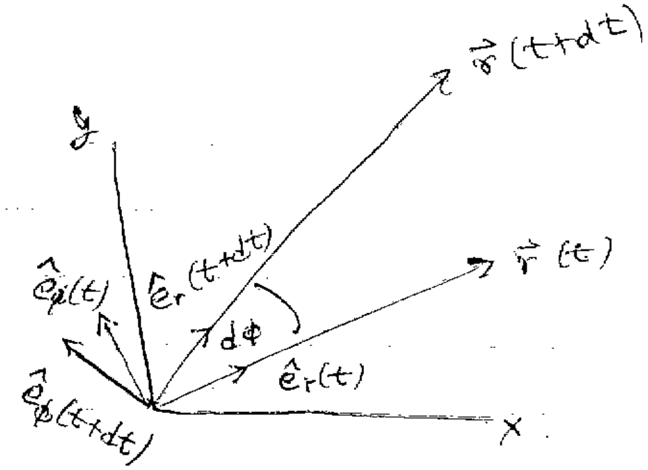
In this coordinate system, $\vec{r}(t) = r\,\hat{e}_r$

velocity $\vec{v}(t) = \dot{\vec{r}}(t) = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r$

Must take time derivative of basis vector $\hat{e}_r$!
Unlike rectangular coordinates, in which $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are fixed and do not vary with time, the basis vectors $\hat{e}_r, \hat{e}_\phi$ vary with time as the particle moves along its trajectory.

What is $\dot{\hat{e}}_r$?

Polar Coordinates are very different from Cartesian coordinates in that now the basis vectors $\hat{e}_r, \hat{e}_\phi$ change direction as the particle moves along $\vec{r}(t)$



$$d\hat{e}_r = \hat{e}_r(t+dt) - \hat{e}_r(t)$$

The point at the tip of the vector $\hat{e}_r$ is rotating as the particle moves from $\vec{r}(t)$ to $\vec{r}(t+dt)$. If it goes through ~~angles~~ angular displacement $\delta\phi$, then by previous discussion

$$\delta\hat{e}_r = \delta\vec{\phi} \times \hat{e}_r$$

where $\delta\vec{\Phi} = \delta\phi\,\hat{e}_z$ points along axis of rotation — in this case $\hat{e}_z$

$$\delta\hat{e}_r = \delta\Phi\;\hat{e}_z \times \hat{e}_r \;=\; \delta\phi\;\hat{e}_\phi$$

So we conclude: $\dfrac{d\hat{e}_r}{dt} = \dfrac{\delta\phi}{\delta t}\,\hat{e}_\phi = \dot{\phi}\,\hat{e}_\phi$

$$\boxed{\dot{\hat{e}}_r = \dot{\phi}\,\hat{e}_\phi}$$

Similarly $\delta\hat{e}_\phi = \delta\vec{\phi} \times \hat{e}_\phi = \delta\phi\;\hat{e}_z \times \hat{e}_\phi = \delta\phi\,(-\hat{e}_r)$

$$\frac{\delta\hat{e}_\phi}{\delta t} = -\frac{\delta\phi}{\delta t}\,\hat{e}_r$$

$$\boxed{\dot{\hat{e}}_\phi = -\dot{\phi}\,\hat{e}_r}$$

velocity: $\vec{v} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$

$$\boxed{\vec{v} = \dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi}$$

acceleration $\vec{a} = \frac{d}{dt}\vec{v} = \frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi)$

$$= \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\phi}\hat{e}_\phi + r\ddot{\phi}\hat{e}_\phi + r\dot{\phi}\dot{\hat{e}}_\phi$$

$$= \ddot{r}\hat{e}_r + \dot{r}\dot{\phi}\hat{e}_\phi + \dot{r}\dot{\phi}\hat{e}_\phi + r\ddot{\phi}\hat{e}_\phi - r\dot{\phi}^2\hat{e}$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{e}_\phi}$$

Example ① Uniform circular motion

$\vec{r} = r\hat{e}_r$ with $r = \text{const}$, $\phi = \omega t$
$\Rightarrow \dot{\phi} = \omega$, $\dot{r} =$

velocity $\vec{v} = (0 + r\omega\hat{e}_\phi)$

$\vec{v} = r\omega\hat{e}_\phi \Rightarrow |\vec{v}| = r\omega$, $\vec{v} \perp \vec{r}$

acceleration $\vec{a} = (0 - r\omega^2)\hat{e}_r + (0 + 0)\hat{e}_\phi$

$\vec{a} = -r\omega^2\hat{e}_r$ radially inwards
$|\vec{a}| = r\omega^2 = \frac{v^2}{r}$

Much easier in polar coordinates!

Example ② A ball sits in a grooved track which rotates about its end with constant angular speed ω.

The side walls of the track exert a normal force on the ball, but the ball is free to slide up and down the track. What is the position of the ball as the track rotates?

Total force on the ball is only the normal force from the walls of the track — this is in the $\hat{e}_\phi$ direction. The net force in the $\hat{e}_r$ direction therefore must vanish
$\Rightarrow$ $\hat{e}_r$ component of $\vec{a}$ must equal zero.

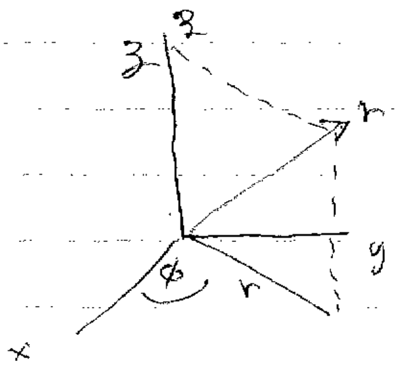
$$a_r = \ddot{r} - r\dot{\phi}^2 = \ddot{r} - r\omega^2 = 0$$

$$\ddot{r} = \omega^2 r \Rightarrow \boxed{r(t) = r(0)\, e^{\omega t}}$$

Note: Although there is no net radial force on the ball, $r(t)$ rapidly increases! If you were an observer sitting on the rotating track, it would look as if there was an outward radial force equal to $mr\omega^2$. This is the "centrifugal force". It is a fictitious force arising from the fact that the observer is in an accelerating (non-inertial) frame of reference. In the lab frame, there is no radial force. The non zero $\ddot{r}$ is just what is needed to cancel out the inward centripetal acceleration so that the net radial acceleration vanishes.

Later in the course we will come accross another fictitious force arising in a rotating frame of reference - this is the "Coriolis force". This comes from the $2\dot{r}\dot{\phi}\,\hat{e}_\phi$ term in $\vec{a}$.

In three dimensions we can generalize to cylindrical coordinates



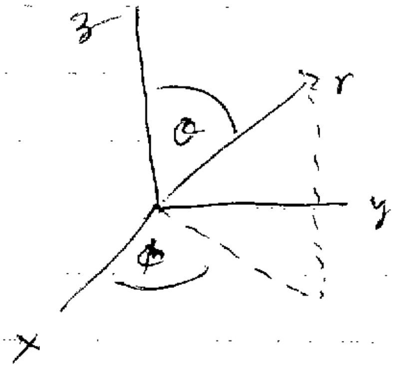
$\vec{r}$ given by coordinates (r, ϕ, z)
(here r is not $|\vec{r}|$, but the length of the projection of $\vec{r}$ into the xy plane

$$\vec{r} = r\hat{e}_r + z\hat{e}_z \qquad \begin{cases} \dot{\hat{e}}_r = \dot{\phi}\hat{e}_\phi \\ \dot{\hat{e}}_\phi = -\dot{\phi}\hat{e}_r \\ \dot{\hat{e}}_z = 0 \end{cases}$$

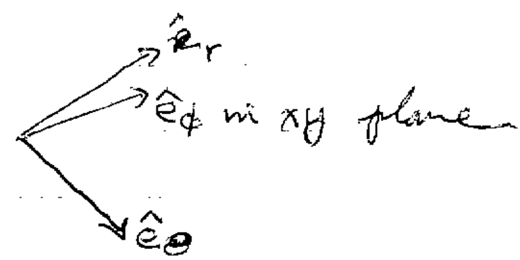
$$\Rightarrow \vec{v} = \dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z$$

$$\vec{a} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z$$

In spherical coordinates $\vec{r}$ given by coordinates r, θ, ϕ



basis vectors



$$\hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z$$

$$\hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y$$

To get time derivatives of $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ we can differentiate the expansions in cartesian coordinates.

$$\dot{\hat{e}}_r = \begin{bmatrix} (\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi)\,\hat{e}_x \\ + (\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi)\,\hat{e}_y \\ - \dot\theta\,\sin\theta\,\hat{e}_z \end{bmatrix} = \dot\theta\,\hat{e}_\theta + \dot\phi\sin\theta\,\hat{e}_\phi$$

$$\dot{\hat{e}}_\theta = \begin{bmatrix} (-\dot\theta\sin\theta\cos\phi - \dot\phi\cos\theta\sin\phi)\,\hat{e}_x \\ (-\dot\theta\cos\theta\sin\phi + \dot\phi\cos\theta\cos\phi)\,\hat{e}_y \\ -\dot\theta\cos\theta\,\hat{e}_z \end{bmatrix} = -\dot\theta\,\hat{e}_r + \dot\phi\sin\theta\,\hat{e}_\phi$$

$$\dot{\hat{e}}_\phi = -\dot\phi\cos\phi\,\hat{e}_x - \dot\phi\sin\phi\,\hat{e}_y \quad = -\dot\phi(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$$

$$\left[\begin{aligned}
\dot{\hat{e}}_r &= \dot{\theta}\,\hat{e}_\theta + \dot{\phi}\sin\theta\,\hat{e}_\phi \\
\dot{\hat{e}}_\theta &= -\dot{\theta}\,\hat{e}_r + \dot{\phi}\sin\theta\,\hat{e}_\phi \\
\dot{\hat{e}}_\phi &= -\dot{\phi}\,(\sin\theta\,\hat{e}_r + \cos\theta)\,\hat{e}_\theta
\end{aligned}\right]$$

spherical coordinates

$$\Rightarrow \vec{v} = \dot{\vec{r}} = \frac{d}{dt}(r\,\hat{e}_r) = \dot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + r\dot{\phi}\sin\theta\,\hat{e}_\phi$$

could similarly compute $\vec{a} = \dot{\vec{v}}$ in spherical coordinates
— but we won't do this.

Vector Calculus

Gradient of a scalar function $f(\vec{r})$ defined by

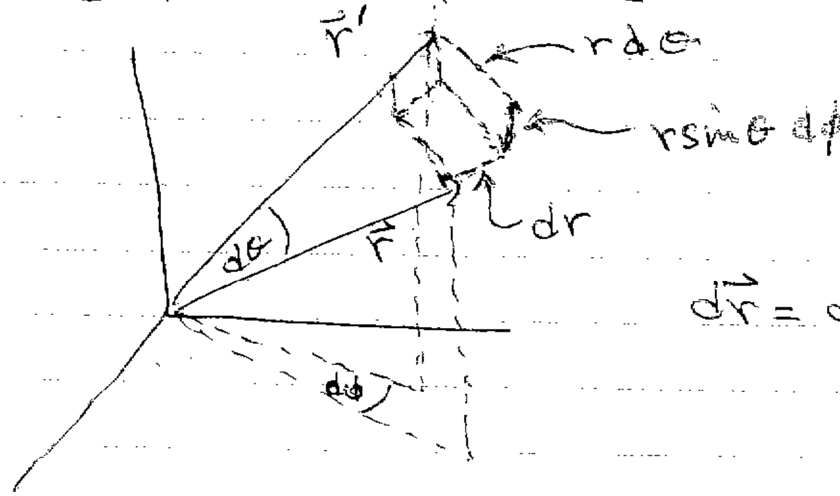
$$df = f(\vec{r}+d\vec{r}) - f(\vec{r}) = \vec{\nabla} f(\vec{r}) \cdot d\vec{r}$$

In Cartesian coordinates $d\vec{r} = dx_1 \hat{e}_1 + dx_2 \hat{e}_2 + dx_3 \hat{e}_3$

$$df = (\vec{\nabla} f)_1 dx_1 + (\vec{\nabla} f)_2 dx_2 + (\vec{\nabla} f)_3 dx_3$$

$$\Rightarrow (\vec{\nabla} f)_i = \frac{\partial f}{\partial x_i} \quad \text{or} \quad \boxed{\begin{aligned}\vec{\nabla} f &= \frac{\partial f}{\partial x_1}\hat{e}_1 + \frac{\partial f}{\partial x_2}\hat{e}_2 + \frac{\partial f}{\partial x_3}\hat{e}_3 \\ &= \sum_i \frac{\partial f}{\partial x_i}\hat{e}_i\end{aligned}}$$

In spherical coordinates



$$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\phi\,\hat{e}_\phi$$

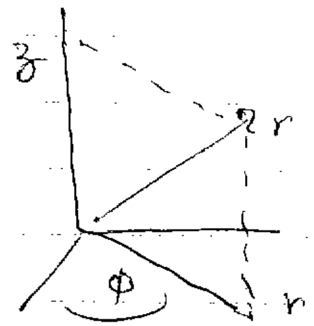
$$df = (\vec{\nabla} f) \cdot (dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\phi\,)\hat{e}_\phi$$

$$\Rightarrow (\vec{\nabla} f)_r = \frac{\partial f}{\partial r}, \quad (\vec{\nabla} f)_\theta = \frac{\partial f}{r\,\partial\theta}, \quad (\vec{\nabla} f)_\phi = \frac{\partial f}{r\sin\theta\,\partial\phi}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{\partial f}{r\,\partial\theta}\hat{e}_\theta + \frac{\partial f}{r\sin\theta\,\partial\phi}\hat{e}_\phi$$

Cylindrical coordinates $\quad d\vec{r} = dr\,\hat{e}_r + r\,d\phi\,\hat{e}_\phi + dz\,\hat{z}$

$$\Rightarrow \vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{\partial f}{r\,\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$



Line integral

From $df = \vec{\nabla} f(\vec{r}) \cdot d\vec{r}$ it follows that

$$f(\vec{r}_2) - f(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_2} df = \int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla} f(\vec{r}) \cdot d\vec{r}$$

line integral along <u>any</u> path from $\vec{r}_1$ to $\vec{r}_2$

$$\oint \vec{\nabla} f(\vec{r}) \cdot d\vec{r} = 0$$

any closed path