Newtons 1st law

A body at rest or in uniform motion continues so unless acted on by external force

Newtons 2nd Law

$$F = m\vec{a} = m\dot{\vec{v}} = \dot{\vec{p}}$$

$\Rightarrow$ [says what a force is

where momentum $\vec{p} = m\vec{v}$

$\Rightarrow$ [defines what mass is

Newton's 3rd Law

If two bodies exert forces on each other these forces are equal in magnitude & opposite in directions. If $\vec{F}_1$ is force on particle one from particle 2, and $\vec{F}_2$ is force on particle 2 from particle 1, then

$$\vec{F}_1 = -\vec{F}_2$$

$$\Rightarrow \quad \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \quad \Rightarrow \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = \text{constant}$$

$$\Rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$$

momentum conservation

(similar argument works for any number of interacting particles)

Mass

In Newton's 2nd law, mass is defined as the ratio of the applied force $\vec{F}$ to the resulting acceleration $\vec{a}$

$$\frac{|\vec{F}|}{|\vec{a}|} \equiv m$$

this is sometimes called the inertial mass – the resistance to acceleration or to change in "inertia"

If have a "standard force" $\vec{F}_0$, say a spring, and a standard mass m_0, then one can determine the mass of any other object relative to m_0. ~~If m is the mass of the object~~ If $\vec{a}_0 = \frac{\vec{F}_0}{m_0}$ is the measured acceleration of the standard mass m_0 when acted on by the standard force $\vec{F}_0$, and if $\vec{a}$ is the measured acceleration of another object when acted on by $\vec{F}_0$, then the mass of the object is

$$m = \frac{F_0}{a} = \frac{a_0 m_0}{a} = \left(\frac{a_0}{a}\right) m_0$$

One well known force is the force of gravity at the surface of the earth. This gravitational force is assumed to be $\vec{F}_g = m_g \vec{g}$ where $g = 9.8 \text{ m/s}^2$ downwards. The gravitational mass m_g is what you measure when you weigh an object ~~on a~~ by compressing a spring in a scale. The acceleration of an object in the gravitational force is $\vec{a} = \frac{\vec{F}_g}{m} = \left(\frac{m_g}{m}\right) \vec{g}$.

The principle of equivalence, was first shown by Galileo. It is that $m = m_g$ for all objects. Inertial mass (the coefficient that determines acceleration in Newton's 2nd Law) equals the gravitational mass (the coefficient that determines the strength of the gravitational force acting on a body)

Inertial Frame of Reference

Reference frame in which Newton's laws of motion are valid — it is a non-accelerating frame of reference.

Since $\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$

if one makes a change of coordinates $\vec{r}' = \vec{r} - \vec{u}t$ to a frame of reference moving with const velocity $\vec{u}$, then

$$\frac{d^2\vec{r}'}{dt^2} = \frac{d^2\vec{r}}{dt^2} = \vec{F}$$

So Newton's laws hold also in the moving frame. This is called Galilean invariance

Work and Energy

The *work* done by a force $\vec{F}$ on a particle as the particle moves from position "1" to position "2" is

$$W_{12} \equiv \int_1^2 \vec{F} \cdot d\vec{r} \qquad \text{line integral}$$

If $\vec{F}$ is the total force on the particle, then

$$\vec{F} \cdot d\vec{r} = (m\vec{a}) \cdot \left(\frac{d\vec{r}}{dt} dt \right) = \left(m \frac{d\vec{v}}{dt} \right) \cdot \left(\vec{v}\, dt \right)$$

$$= \frac{m}{2} \frac{d}{dt} (v^2)\, dt \qquad \cancel{\frac{dv^2}{dt} = 2\vec{v}\cdot\dot{\vec{v}}}$$

Since $\displaystyle \frac{d}{dt} v^2 = \frac{d}{dt} \left(\sum_i v_i^2 \right) = \sum_i 2 v_i \frac{dv_i}{dt} = 2 \vec{v} \cdot \frac{d\vec{v}}{dt}$

$$\vec{F} \cdot d\vec{r} = \frac{d}{dt} \left(\frac{m}{2} v^2 \right) dt = d \left(\frac{m v^2}{2} \right)$$

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 d \left(\frac{m v^2}{2} \right) = \frac{m v_2^2}{2} - \frac{m v_1^2}{2}$$

Define $T = \frac{1}{2} m v^2$ *kinetic energy*

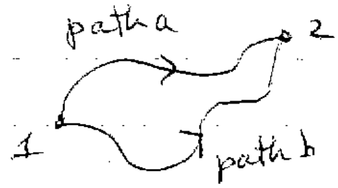
Work done = change in kinetic energy

$$W_{12} = T_2 - T_1$$

If $W_{12} > 0$, ie work done on particle is positive then $T_2 > T_1 \Rightarrow$ kinetic energy increases.

Conservative forces

In general $\int_1^2 \vec{F} \cdot d\vec{r}$ may depend on the path taken in going from "1" to "2".



For certain forces, however, this integral is independent of the path. Such forces are called conservative forces.

Conservative forces are forces that can be written as the gradient of a scalar field U.

$$\vec{F} = -\vec{\nabla} U$$

then $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = -\int_1^2 \vec{\nabla} U \cdot d\vec{r} = -\int_1^2 dU$

$$W_{12} = -\left(U(r_2) - U(r_1) \right) = U_1 - U_2$$

U is called the potential energy of the force $\vec{F}$.

For conservative force, $\oint \vec{F}\cdot d\vec{r} = 0$

closed loop

Stokes theorem: $\oint \vec{F}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$

$$\Rightarrow \vec{\nabla}\times\vec{F} = 0 \quad \text{for conservative force}$$

can check that it is always true that

$\vec{\nabla}\times(\vec{\nabla}U) = 0$ for any scalar function U.

Now $W_{12} = T_2 - T_1 = U_1 - U_2$

$$\Rightarrow T_2 + U_2 = T_1 + U_2 \equiv E$$ total mechanical energy = kinetic + potential

$\Rightarrow$ Conservation of mechanical energy when forces are conservative

Example gravity at earth's surface $\vec{F}_g = m\vec{g} = -mg\,\hat{e}_z$

where $\hat{e}_z$ is normal to surface [illegible]

$$\vec{F} = -\vec{\nabla}U \quad \text{where} \quad U = mgz$$

Note, potential energy is not absolute - can always add any constant to the function U, without changing the force $\vec{F} = -\vec{\nabla}U$.

Also, T is not absolute, since can always transform to another moving inertial frame, ~~Energy conserved~~ ~~motion in~~ which would change velocity $\vec{v}$.

Conservation Theorems

Linear momentum

When total force $\vec{F} = 0$ then

$$\vec{F} = \frac{d\vec{p}}{dt} = 0 \quad \text{and} \quad \vec{p} = \text{constant} \quad \text{momentum conserved}$$

If the component of $\vec{F}$ in any particular direction $\hat{e}$ vanishes then

$$\vec{F}\cdot\hat{e} = \frac{d\vec{p}}{dt}\cdot\hat{e} = 0 \quad \Rightarrow \vec{p}\cdot\hat{e} = \text{constant}$$

component of momentum in direction $\hat{e}$ is conserved

Example: in gravitational field $\vec{F}_g$, components of $\vec{F}$ parallel to surface of earth are zero $\Rightarrow$ horizontal components of momentum are conserved.

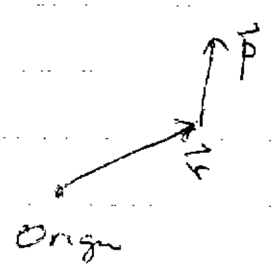
Angular momentum

Angular momentum of a particle with respect to an origin from which the position $\vec{r}$ is measured is defined to be

$$\vec{L} = \vec{r} \times \vec{p}$$

the torque with respect to the same origin is

$$\vec{\tau} = \vec{r} \times \vec{F}$$



$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Consider

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

But $\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0$ as $\vec{v} \times \vec{v} = 0$

So

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{\tau}$$

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$$ angular version of $\vec{F} = \frac{d\vec{p}}{dt}$

If component of torque in a particular direction $\hat{e}$ vanishes, $\vec{\tau} \cdot \hat{e} = 0$, then $\hat{e} \cdot \frac{d\vec{L}}{dt} = \frac{d}{dt}(\hat{e} \cdot \vec{L}) = 0$

then the component of $\vec{L}$ in the direction $\hat{e}$ is conserved

$$\vec{L} \cdot \hat{e} = \text{constant}$$

Energy — Generalize to time dependent potential energy

$E = T + U =$ ~~constant~~ Total mechanical energy ~~energy conserved~~

U is potential energy for any conservative forces

$$\Rightarrow \frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

$$W = \vec{F} \cdot d\vec{r} = dT \Rightarrow \frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$$\frac{dU(\vec{r}(t),t)}{dt} = \sum_i \frac{\partial U}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial U}{\partial t} = (\vec{\nabla} U) \cdot \frac{d\vec{r}}{dt} + \frac{\partial U}{\partial t}$$

$$\Rightarrow \frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} + (\vec{\nabla} U) \cdot \frac{d\vec{r}}{dt} + \frac{\partial U}{\partial t}$$

$$= \left(\vec{F} + \vec{\nabla} U\right) \cdot \frac{d\vec{r}}{dt} + \frac{\partial U}{\partial t}$$

If total force is conservative then $\vec{F} = -\vec{\nabla} U$ then

$$\frac{dE}{dt} = \frac{\partial U}{\partial t}$$

If potential energy U is not an explicit function of time t, then $\frac{\partial U}{\partial t} = 0 \Rightarrow \frac{dE}{dt} = 0$ and mechanical energy is conserved.

In earlier derivation, it should have been

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = -\int_1^2 \vec{\nabla} U(\vec{r}(t),t) \cdot \frac{d\vec{r}}{dt}\, dt = -\int_1^2 \left[\frac{dU}{dt} - \frac{\partial U}{\partial t}\right] dt$$

$$W_{12} = -\int_1^2 dU + \int_1^2 \frac{\partial U}{\partial t}\, dt = U(r_1,t_1) - U(r_2,t_2) + \int_1^2 \frac{\partial U}{\partial t}\, dt$$