Applications of Newton's Laws

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$$

solve second order differential equation
If one knows the force $\vec{F}(t)$ as function of time then can solve equation of motion by integration, if given initial conditions $\vec{r}(0)$ and $\vec{v}(0)$

$$\vec{v}(t) = \int_0^t dt' \frac{\vec{F}(t')}{m} + \vec{v}(0)$$

$$\vec{r}(t) = \int_0^t dt' \, \vec{v}(t') + \vec{r}(0)$$

In general, not so simple since $\vec{F}$ may depend on position of particle $\vec{r}$.

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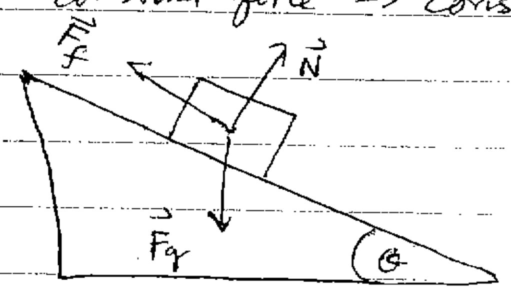
Constant force problems

constant force ⇒ constant acceleration



$\vec{N}$ = normal force
$\vec{F}_g$ = gravity
$\vec{F}_f$ = friction

$$\vec{F} = \vec{F}_g + \vec{N} + \vec{F}_f = m\vec{a}$$

Take components perpendicular + parallel to surface

As a simple case, consider a constant force $\vec{F}$.

$$\vec{v}(t) = \int_0^t dt' \, \frac{\vec{F}}{m} + \vec{v}(0) = \frac{\vec{F}}{m} t + v(0)$$

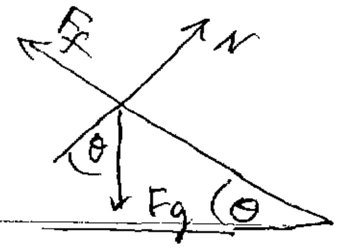
$$\vec{r}(t) = \int_0^t dt' \left(\frac{\vec{F}}{m} t + \vec{v}(0) \right) + \vec{r}(0)$$

$$\vec{r}(t) = \frac{1}{2} \frac{\vec{F}}{m} t^2 + \vec{v}(0)\, t + \vec{r}(0)$$

above is equation of motion at constant acceleration $\vec{a} = \frac{\vec{F}}{m}$ with initial conditions $\vec{v}(0)$, $\vec{r}(0)$

For gravity $\vec{F} = -g\,\hat{e}_z$

$$\vec{r}(t) = -\frac{1}{2} g t^2 \,\hat{e}_z + \vec{v}(0) t + \vec{r}(0)$$



perp: $N - F_g \cos\theta = 0$ no accelleration in $\perp$ direction

$N = F_g \cos\theta = mg\cos\theta$

parallel: $F_g \sin\theta - F_f = ma$ friction $F_f = \mu N$

$mg\sin\theta - \mu N = ma$

$mg\sin\theta - \mu mg\cos\theta = ma$

$mg(\sin\theta - \mu\cos\theta) = ma$

$$a = g(\sin\theta - \mu\cos\theta)$$

For static case where $a=0$, $\mu = \mu_s$ coefficient of static friction and $F_f^{max} = \mu_s N$, then

$$\sin\theta_{max} - \mu_s \cos\theta_{max} = 0$$

determines max angle of incline θ_{max} before block sly

$$\tan\theta_{max} = \mu_s$$

For case where $a > 0$, $\mu = \mu_k$ coefficient of kinetic frict

$$a = g(\sin\theta - \mu_k \cos\theta)$$

In general $\mu_k < \mu_s$

Linear Restoring Force

$F(x) = -kx$ Hook's Law ex: stretched spring

$F = ma \Rightarrow m\frac{d^2x}{dt^2} = -kx$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

define $\omega_0^2 = \frac{k}{m}$

$$\frac{d^2x}{dt^2} = -\omega_0^2 x$$ single harmonic oscillator

General solution has the form

$$x(t) = A\sin(\omega_0 t - \delta)$$

$$= A\cos\delta\,\sin(\omega_0 t) - A\sin\delta\,\cos(\omega_0 t)$$

oscillations with angular freq $\omega_0 = \sqrt{\frac{k}{m}}$

Note $F(x) = -kx = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right)$

$\Rightarrow F(x)$ is conservative with potential $U(x) = \frac{1}{2}kx^2$

Total mechanical energy

$E = T + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ is conserved

General Conservative forces in one dimension

$$-\int_1^2 \vec{F}\cdot d\vec{r} = U_2 - U_1$$

$$E = T + U = \tfrac{1}{2}mv^2 + U(x) \quad \text{is conserved}$$

$$\Rightarrow \quad v^2 = \frac{2}{m}\left[E - U(x)\right]$$

$$\Rightarrow \quad v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\,|E - U(x)|}$$

$$\Rightarrow \quad \int dt = \int \frac{\pm\, dx}{\sqrt{\frac{2}{m}\,|E - U(x)|}}$$

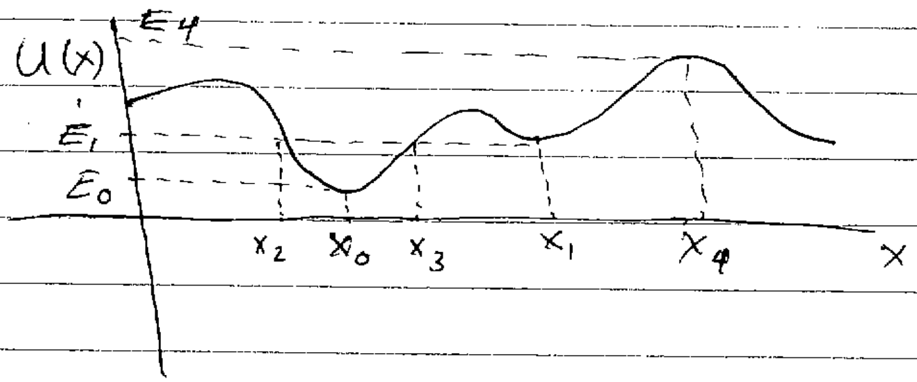
$$t - t_0 = \int_{x_0}^{x} \frac{\pm\, dx'}{\sqrt{\frac{2}{m}\,|E - U(x')|}} \qquad \text{where } x(t_0) = x_0$$

If we know $U(x)$, we can in principle do the integration and get $t - t_0 = g(x)$

function we get after doing integral

we can then solve for $x(t) = g^{-1}(t - t_0)$

In general we can understand what to expect for such motion.



points x_0 and x_1 are <u>stable equilibria</u>.
At these points, $F = -\frac{dU}{dx} = 0$
$\Rightarrow$ no force on particle, so particle at rest
<u>Also</u>: if perturb particle's position $x_0 \to x_0 + \delta$, the resulting force F pushes particle <u>back</u> to x_0

point x_4 is <u>unstable equilib</u>
Here $F = -\frac{dU}{dx} = 0$ also, so particle at rest. But if perturb $x_4 \to x_4 + \delta$, the resulting force pushes the particle away from x_4.

If the particle has energy E_1, it may either be at the stable point x_1, or it may oscilate between points x_2 and x_3. For such oscilation, $E = U$ and hence $T = 0$ and so $v = 0$, when $x = x_2$ or x_3. These are the "turning points" where velocity vanishes and particles motion reverses.
At $x = x_0$, U is minimum so T and hence v

is maximum.

If particle has energy E_4, it will come in from left, slow down and stop when reaches x_4, then reverse directions and travel back to left.

Near a minimum of $U(x)$, say at x_0, one can always write

Taylor expansion: $U(x) \simeq U(x_0) + \frac{1}{2} U''(x_0)(x-x_0)^2$

2nd deriva

$$\Rightarrow \quad F = -\frac{dU}{dx} = -U''(x_0)(x-x_0) \qquad U'' \equiv \frac{d^2U}{dx^2}$$

Newtons 2nd law is then

$$m\frac{dx^2}{dt^2} = -U''(x_0)(x-x_0)$$

let $x' = x - x_0$, then $dx' = dx$

$$m\frac{dx'^2}{dt^2} = -U''(x_0)\,x'$$

since x_0 is a min, $U''(x_0) > 0$

$$\Rightarrow \quad \frac{dx'^2}{dt^2} = -\frac{U''(x_0)}{m}\,x'$$

x' undergoes simple harmonic motion at angular frequency $\omega_0 = \sqrt{\frac{U''(x_0)}{m}}$

curvature of $U(x)$ at minimum determines ω_0

velocity dependent forces

A particle moving through a fluid (such as air) experiences a viscous drag force that can often be approximated as

$$\vec{F}_D(v) = -k v^n \hat{v}$$

k is positive
$\hat{v}$ is direction of $\vec{v}$

Experimentally, for small objects moving at at low velocities (~ 24 m/s $= 80$ ft/s) in air $n \simeq 1$. For higher velocities (but lower than speed of sound ~ 330 m/s $\simeq 1{,}100$ ft/s) $n \simeq 2$.

In this latter regime, for air we have

$$\vec{F}_D(v) = -\tfrac{1}{2} C_D \rho A v^2 \hat{v}$$

where C_D is a dimensionless drag coefficient, ρ is the density of air, A is cross section of object $\perp$ to direction of motion.

More generally for air, a good approx is

$$\vec{F}_D(v) \simeq -(c_1 v + c_2 v^2)\, \hat{v}$$

where for spherical objects of diameter D,
$c_1 = 1.55 \times 10^{-4} D$ and $c_2 = 0.22 D^2$ in MKS units

Example

① vertical fall, low velocities $F_D \sim v$ $(c_1 \gg c_2 v$

$$m\frac{dv}{dt} = F = -mg - c_1 v$$

separate variables

$$\int_{v_0}^{v} \frac{m\,dv'}{-mg - c_1 v'} = \int_0^t dt' \qquad v_0 = v(0)$$

$$t = -\frac{m}{c_1}\left[\ln\left(v + \frac{m}{c}g\right)\right]_{v_0}^{v} = -\frac{m}{c_1}\ln\left(\frac{v + \frac{m}{c}g}{v_0 + \frac{m}{c}g}\right)$$

define $\tau = \frac{m}{c_1}$ solve for v in terms of t

$$e^{-t/\tau} = \frac{v + \frac{m}{c}g}{v_0 + \frac{m}{c}g}$$

$$v = \left(v_0 + \frac{m}{c}g\right)e^{-t/\tau} - \frac{m}{c}g$$

as $t \to \infty$, velocity approaches the value

$v_\infty = -\frac{m}{c}g$ known as the "terminal" velocity

For an object falling from rest, $v_0 = 0$

$$v = v_\infty\left(1 - e^{-t/\tau}\right)$$

For a small raindrop with $D = 0.5$ mm

we get $c_1 = (1.55 \times 10^{-4})(0.5 \times 10^{-3}) = 0.775 \times 10^{-7}$

$$m \simeq \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 \cdot \left(\frac{kg\ 10^{-3}}{10^{-6}}\right) \simeq 6.54 \times 10^{-8}\ kg$$

$$\rho = 1 g/cm^3$$

$$\tau = \frac{m}{c_1} = \frac{6.54 \times 10^{-8}}{0.775 \times 10^{-7}} = 0.84 \text{ sec}$$

$$|v_\infty| = \frac{mg}{c_1} = (0.84 \text{ sec})(9.8\ m/s^2) = 8.3\ m/s$$

So rain drops reach terminal velocity of 8.3 m/s in less than one second!

Compare this the the speed of a raindrop if there was no viscous drag. Then $v(t) = -gt + v(0)$

$$x(t) = -\frac{1}{2}gt^2 + v(0)t + x(0)$$

If dropped from $x(0) = h$ at $v(0) = 0$, then hits ground in a time t given by

$$0 = -\frac{1}{2}gt^2 + h$$

$$t = \sqrt{2h/g}$$

speed at this time is $|\vec{v}| = gt = \sqrt{2hg}$

(could have gotten above by energy conservation)

$$\frac{1}{2}mv^2 = mgh \Rightarrow v = \sqrt{2gh}$$

For a rain drop falling from 2000 m, factor 24 times bigger than v_∞

$$v = \sqrt{2(9.8)(2000)}\ m/s = 198\ m/s$$

$$= 198\left(\frac{m}{s}\right)\cdot\left(\frac{100\ cm}{m}\right)\left(\frac{inch}{2.5\ cm}\right)\left(\frac{ft}{12\ in}\right)\left(\frac{mi}{5280\ ft}\right)\cdot\left(\frac{3600\ s}{hr}\right)$$

$$= 450\ mph\ !$$

$v_\infty = \frac{mg}{c_1}$ terminal speed $\sim m$
increases for larger rain drops or hail stones

(2) vertical fall, higher velocities $F_D \sim v^2$ $(c_1 << c_2)$

in general: $m\frac{dv}{dt} = -mg \pm c_2 v^2$ where $\left\{\begin{array}{l} + \text{ is for falling} \\ - \text{ is for rising} \end{array}\right.$
since $\vec{F}_D$ is opposite to $\vec{v}$

falling

$$m\frac{dv}{dt} = -mg + c_2 v^2$$

$$\frac{m}{c_2}\int_{v_0}^{v}\frac{dv'}{v'^2 - \frac{m}{c_2}g} = \int_0^t dt' = t$$

look up integral in hand book

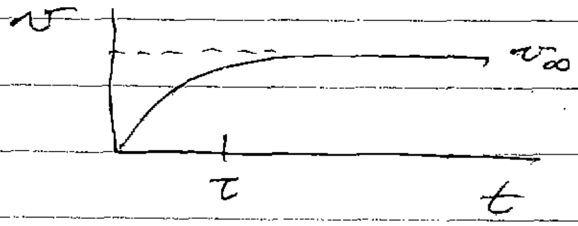
$$t = \tau\left(\tanh^{-1}\left(\frac{v_0}{v_\infty}\right) - \tanh^{-1}\left(\frac{v}{v_\infty}\right)\right)$$

where $\tau = \sqrt{\frac{m}{c_2 g}}$ and $v_\infty = \sqrt{\frac{mg}{c_2}}$, so $\tau = \frac{v_\infty}{g}$

for an object dropped from rest, $v_0 = 0$

$$t = -\tau \tanh^{-1}\left(\frac{v}{v_\infty}\right)$$

$$v(t) = v_\infty \tanh\left(\frac{t}{\tau}\right)$$



ex: for $m = 0.6$ kg basket ball with $D = 0.25$ m

$$c_2 = (0.22)(.25)^2 = 0.0138$$

$$\tau = \sqrt{\frac{m}{c_2 g}} = \sqrt{\frac{0.6}{(0.0138)(9.8)}} = 2.1\ \text{s}$$

$$v_\infty = \tau g = (0.67)(9.8)\ \text{m/s} = 20\ \text{m/s}$$

③ Projectile motion in air

For simplicity assume $F_D = -kmv$
(although it is not really true)

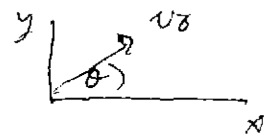
horizontal $\quad m\dfrac{d^2x}{dt^2} = -km\dfrac{dx}{dt}$

vertical $\quad m\dfrac{d^2y}{dt^2} = -km\dfrac{dy}{dt} - mg$

Solve in horizontal direction

horizontal velocity dec[reases] to zero

$$\frac{dv_x}{dt} = -kv_x \Rightarrow v_x(t) = v_{x0}e^{-kt}$$



where $v_{0x} = v_0 \cos\theta$

horizontal distance traveled from $x(0) = 0$

$$x(t) = \int_0^t dt' \, v_x(t') = \int_0^t dt' \, v_{x0} e^{-kt}$$

$$= \frac{v_{x0}}{k}\left(1 - e^{-kt}\right)$$

The range of the projectile is determined by solving for $y(t)$ (can't do analytically - need perturbation or numerical solution - see text)

Finding time t_f where $y = 0$ - this is time projectile hits ground

Range is then $x(t_f)$.

But note, although we can't easily do above calculation, we can say that the range can never be greater than $\frac{v_{x0}}{k}$

Compare this to case where there is no drag force

Then $v_x(t) = v_{x0}$ is <u>constant</u>

See Fig 2-8 of text for graph of more complete computation of range.