Oscillations

Simple harmonic oscillator - one dimension

$$F = -kx$$ ← true for any conservative force with potential $U(x)$, for oscillations about a potential minimum x_0 then $k \equiv \frac{d^2U(x_0)}{dx^2}$

$$m\ddot{x} = -kx \quad \Rightarrow \quad \ddot{x} = -\left(\frac{k}{m}\right)x$$

define $\omega_0^2 = \frac{k}{m} \quad \Rightarrow \quad \ddot{x} = -\omega_0^2 x$

General solution $\quad x(t) = A\sin(\omega_0 t - \delta)$

or $\quad A\cos(\omega_0 t - \phi) \quad$ where $\phi = \delta +$

initial conditions $\begin{cases} x(0) = A\cos(\phi) \\ \dot{x}(0) = +\omega_0 A\sin(\phi) \end{cases}$

Potential $\quad U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t - \phi)$

Kinetic $\quad T = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\sin^2(\omega_0 t - \phi)\cdot\omega_0^2$

$$E = U + T = \frac{1}{2}A^2\left(k\cos^2(\omega_0 t - \phi) + m\omega_0^2\sin^2(\omega_0 t - \phi)\right)$$

but $m\omega_0^2 = k$

$$E = \frac{1}{2}A^2\left(k\cos^2(\omega_0 t - \phi) + k\sin^2(\omega_0 t - \phi)\right)$$
$$= \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2 A^2$$

E is constant in time

we know this had to be true since total mechanical energy is conserved for a conservative force.

Complex exponential

$$\ddot{x} = -\omega_0^2 x$$

guess exponential solution $\quad x(t) = x_0 e^{\alpha t}$

substitute in

$$\alpha^2 x_0 e^{\alpha t} = -\omega_0^2 x_0 e^{\alpha t}$$

solution if

$$\alpha^2 = -\omega_0^2$$

$$\Rightarrow \alpha = \pm i\omega_0$$

$$x(t) = A e^{i\omega_0 t} + B e^{-i\omega_0 t}$$

complex exponential obeys algebraic rules of ordinary exponential, ie $\quad e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$

$$(e^{i\theta})^n = e^{in\theta} \quad \text{etc.}$$

One can show that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

expand in the Taylor series of both sides, and show real parts and imaginary parts of both sides are equal.

check: $\quad e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

Real part $\rightarrow \cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

Imaginary part $\Rightarrow \sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$\Big\}$ double angle formulas

The solution above

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

where A and B can in general be complex numbers, is a complex function. The physical solution must be real. So

$$x(t) = \mathrm{Re}\left[Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right]$$

$$= \mathrm{Re}[A]\cos\omega_0 t + \mathrm{Re}[B]\cos\omega_0 t - \mathrm{Im}[A]\sin\omega_0 t - \mathrm{Im}[B]\sin\omega_0 t$$

$$= \left(\mathrm{Re}[A] + \mathrm{Re}[B]\right)\cos\omega_0 t - \left(\mathrm{Im}[A] - \mathrm{Im}[B]\right)\sin\omega_0 t$$

$$x(t) = C\cos\omega_0 t + D\sin\omega_0 t \qquad C = \mathrm{Re}[A+B]$$
$$D = -\mathrm{Im}[A+B]$$

$$\equiv \bar{A}\cos(\omega_0 t - \phi)$$

$$\text{where} \quad \left.\begin{aligned} \bar{A}\cos\phi &= C \\ \bar{A}\sin\phi &= D \end{aligned}\right\} \Rightarrow \begin{aligned} C^2 + D^2 &= \bar{A} \\ \frac{D}{C} &= \tan\phi \end{aligned}$$

so complex exponential solution reduces to the same form as our earlier trigonometric solutions.

ω_0 = angular frequency

$\nu_0 = \frac{\omega_0}{2\pi}$ = frequency

$T_0 = \frac{1}{\nu_0} = \frac{2\pi}{\omega_0}$ = period we had that the total energy

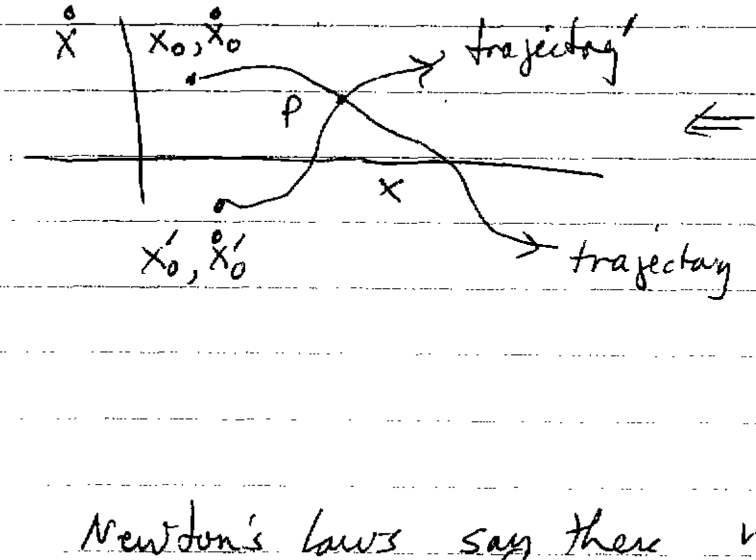
$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \text{constant}$$

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = E$$

$$\Rightarrow \frac{\dot{x}^2}{\left(\frac{2E}{m}\right)} + \frac{x^2}{\left(\frac{2E}{k}\right)} = 1$$

equation of an ellipse in the $(x, \dot{x})$ plane ≡ "phase spac

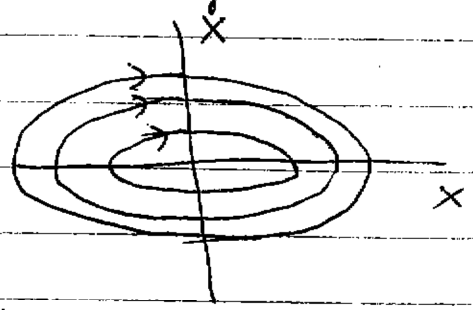
By Newton's laws, specifying an initial position and velocity, $x(0)$, $\dot{x}(0)$, is sufficient to solve for resulting particle trajectory (assuming force is known function of x. ⇒ If plot trajectories of particle in phase space for various initial conditions, the trajectories should never cross!



⇐ this cant happen! If starte
system with initial conditi
at the intersection point P
where would the particle go
would it follow trajectory
or trajectory'?

Newton's laws say there must be a ~~unique~~ unique trajectory for each starting point. Therefore trajectories in phase space can never cross!

trajectories of harmonic oscillators are ellipse



each curve represents a trajectory at a different energy

"phase plot"

<u>Two dimensional harmonic oscillator</u>

$$m\ddot{x} = -k_x x \Rightarrow \ddot{x} = -\omega_x x \qquad \omega_x = \frac{k_x}{m}$$
$$m\ddot{y} = -k_y y \qquad \ddot{y} = -\omega_y y \qquad \omega_y = \frac{k_y}{m}$$

when $\omega_x = \omega_y = \omega$

solutions are

$$x(t) = A\cos(\omega_0 t - \alpha)$$
$$y(t) = B\cos(\omega_0 t - \beta)$$

$$\Rightarrow \cancel{\frac{x^2}{A^2} = \cos^2(\omega_0 t - \alpha)}$$

$$\cancel{\frac{y^2}{B^2} = \cos^2(\omega_0 t - \beta)}$$

Let $\beta = \alpha - \delta$ defines δ

Let $\theta(t) \equiv \omega_0 t - \alpha$

then $x = A\cos\theta$

$$y = B\cos(\theta + \delta) = B\cos\theta\cos\delta - B\sin\theta\sin\delta$$

$$y = B\frac{x}{A}\cos\delta - B\sqrt{1 - \left(\frac{x}{A}\right)^2}\sin\delta$$

$$Ay - Bx\cos\delta = -B\sqrt{A^2 - x^2}\,\sin\delta$$

square both sides

$$A^2y^2 + B^2x^2\cos^2\delta - 2ABxy\cos\delta = B^2(A^2 - x^2)\sin^2\delta$$

$$A^2y^2 + B^2x^2\cos^2\delta + B^2x^2\sin^2\delta - 2ABxy\cos\delta = B^2A^2\sin^2\delta$$

$$A^2y^2 + B^2x^2 - 2ABxy\cos\delta = (AB\sin\delta)^2$$

equation for trajectory in the xy plane.

If $\delta = \frac{\pi}{2}$ then above becomes

$$A^2y^2 + B^2x^2 = A^2B^2 \Rightarrow \left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 = 1 \quad \text{ellipse}$$

If $\delta = 0$ then above gives

$$A^2y^2 + B^2x^2 - 2ABxy = 0 \Rightarrow (Ay - Bx)^2 = 0$$

$$\Rightarrow y = \frac{Bx}{A} \quad \text{straight l.}$$

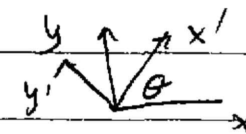
For $A = B$, general δ, the trajectory in the xy plane is always an ellipse oriented along a diagonal – see Fig 3-1 in the text.

For $A = B$ we have

$$x^2 + y^2 - 2xy\cos\delta = A^2 \sin^2\delta$$

Make transformation to ~~dia~~ rotated coordinates x', y'

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$



$\theta = \frac{\pi}{4}$ for diagonal

$$\Rightarrow \left.\begin{aligned} x &= \cos\theta\, x' - \sin\theta\, y' \\ y &= \sin\theta\, x' + \cos\theta\, y' \end{aligned}\right\} \text{ for } \theta = \frac{\pi}{4} \Rightarrow \left\{\begin{aligned} x &= \frac{1}{\sqrt{2}} x' - \frac{1}{\sqrt{2}} y' \\ y &= \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y' \end{aligned}\right.$$

$$x^2 + y^2 - 2xy\cos\delta = \left(\tfrac{1}{2}x'^2 + \tfrac{1}{2}y'^2 - x'y^2\right) + \left(\tfrac{1}{2}x'^2 + \tfrac{1}{2}y'^2 + x'y'\right.$$

$$- 2\cos\delta\left(\tfrac{1}{2}x'^2 - \tfrac{1}{2}y'^2 + \tfrac{1}{2}x'y' - \tfrac{1}{2}x'y'\right)$$

$$= x'^2(1 - \cos\delta) + y'^2(1 + \cos\delta) = A^2 \sin^2\delta$$

$$\Rightarrow \frac{x'^2}{1 + \cos\delta} + \frac{y'^2}{1 - \cos\delta} = \frac{A^2 \sin^2\delta}{1 - \cos^2\delta} = A^2$$

$$\Rightarrow \frac{x'^2}{A^2(1 + \cos\delta)} + \frac{y'^2}{A^2(1 - \cos\delta)} = 1$$ ellipse in x', y' coordinates

for $\delta = \frac{\pi}{2}$, $\cos\delta = 0$, we get a circle of radius A

for $\delta = 0$ we get $y'^2(2) = 0$

$$y' = 0$$

trajectory is straight line along x' axis two dimensional harmonic oscillator –

$\omega_x \neq \omega_y$

solutions $\quad x(t) = A\cos(\omega_x t - \alpha)$

$\qquad\qquad y(t) = B\cos(\omega_y t - \beta)$

trajectories are called "Lissajous" figures.

when $\dfrac{\omega_x}{\omega_y}$ is a rational fraction

then the trajectory in the xy plane is a closed curve, ie the trajectory is periodic in time –

when $\dfrac{\omega_x}{\omega_y}$ is irrational the trajectory is not periodic, the curve is not closed, but as time increases it sweeps out the entire rectangle $x \in [-A, A]$, $y \in [-B, B]$ See fig 3-2, 3-3 in Text.

Damped free oscillations

For simple harmonic oscillator, $\ddot{x} = -\omega_0^2 x$, solution keeps oscillating in time with no decay in amplitude.

In most physical systems, there is some dissipation mechanism that causes the energy stored in the oscillator to leak out over time, and for the amplitude of the oscillations to decay. We can model such a damping force by $F_D = -b\dot{x}$

damped harmonic oscillator:

damping force

restoring force = Hook's law

$$m\ddot{x} = -b\dot{x} - kx$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \qquad \omega_0^2 = \frac{k}{m}, \ \beta = \frac{b}{2m}$$

To get the solution substitute in the guess

$$x(t) = x_0 e^{i\omega t}$$

(at the end, we will take the real part of this complex solution to get the physical solution)

$$(i\omega)^2 x_0 e^{i\omega t} + (i\omega)(2\beta) x_0 e^{i\omega t} + \omega_0^2 x_0 e^{i\omega t} = 0$$

$$\Rightarrow \ -\omega^2 + 2i\beta\omega + \omega_0^2 = 0$$

$$\omega^2 - 2i\beta\omega - \omega_0^2 = 0$$

we have a solution if

$$\omega = i\beta \pm \sqrt{(i\beta)^2 + \omega_0^2}$$

$$\omega = i\beta \pm \sqrt{\omega_0^2 - \beta^2}$$

There are three cases to consider

① $\omega_0 > \beta \Rightarrow$ Underdamped oscillations

define $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ real number

then $\omega = i\beta \pm \omega_1$

general complex solution is

$$x(t) = A e^{i(i\beta + \omega_1)t} + B e^{i(i\beta - \omega_1)t}$$

$$= A e^{-\beta t} e^{i\omega_1 t} + B e^{-\beta t} e^{-i\omega_1 t}$$

exact same form as complex solution for undamped oscillator except for two important differences

1. amplitude decays a $e^{-\beta t} \Rightarrow$ decay time $\tau = 1/\beta$
2. freq of oscillation ω_1 is shifted down from undamped freq ω_0

The physical solution is given by the real part of the above. As for the undamped oscillator, we can write this in the form

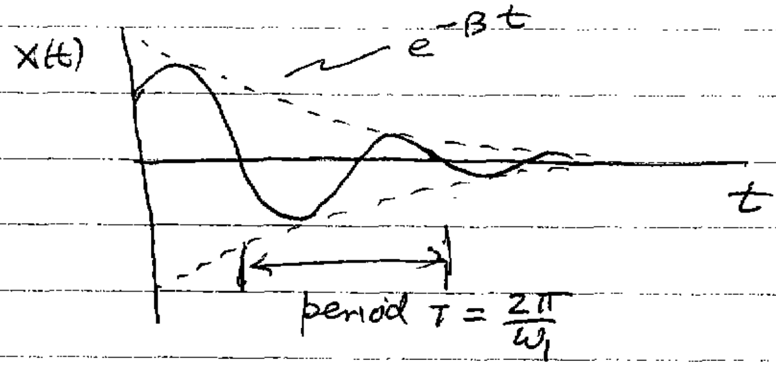
$$x(t) = \bar{A} e^{-\beta t} \cos(\omega_1 t - \phi)$$

where $\bar{A}$ and ϕ are determined by the initial condition.

$$① \quad x(0) = \bar{A} \cos(\phi)$$

velocity: $\dot{x}(t) = -\beta \bar{A} e^{-\beta t} \cos(\omega_1 t - \phi) - \omega_1 \bar{A} e^{-\beta t} \sin(\omega_1 t - \phi)$

$$\Rightarrow ② \quad \dot{x}(0) = -\beta \bar{A} \cos\phi + \omega_1 \bar{A} \sin\phi$$



The exponential decay of the amplitude is related to the loss of energy in the oscillator.

$$E = T + U = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2$$

$$E = \frac{1}{2}m\Big(\beta^2 \bar{A}^2 e^{-2\beta t}\cos^2(\omega_1 t - \phi) + \omega_1^2 \bar{A}^2 e^{-2\beta t}\sin^2(\omega_1 t - \phi)$$

$$+ 2\beta\omega_1 \bar{A}^2 \cos(\omega_1 t - \phi)\sin(\omega_1 t - \phi)\Big)$$

$$+ \frac{1}{2}k\left(\bar{A}^2 e^{-2\beta t}\cos^2(\omega_1 t - \phi)\right)$$

If we consider the <u>average</u> energy $\langle E \rangle$, averaged over one cycle of oscillation, then since the average over one cycle $\langle \cos^2(\omega_1 t - \phi)\rangle = 1/2$

and $\langle \cos(\omega_1 t - \phi)\sin(\omega_1 t - \phi)\rangle = \frac{1}{2}\langle \cos(2\omega_1 t - 2\phi)\rangle =$

we get

use $\omega_1^2 = \omega_0^2 - \beta^2$

$$\langle E \rangle = \frac{1}{4}m(\beta^2 + \omega_1^2)\bar{A}^2 e^{-2\beta t} + \frac{1}{4}k\bar{A}^2 e^{-2\beta t}$$

$$= \frac{1}{4}(m\omega_0^2 + k)\bar{A}^2 e^{-2\beta t} = \frac{1}{2}k\bar{A}^2 e^{-2\beta t}$$

since $m\omega_0^2 = k$

$$\langle E \rangle = \frac{1}{2}k\bar{A}^2 e^{-2\beta t}$$

$$= E_0\, e^{-2\beta t}$$

initial energy

decay time for energy is $\frac{1}{2\beta}$ half as big as decay time for amplitude — since energy $\propto$ (amplitude)^2

② $\omega_0 < \beta \Rightarrow$ <u>overdamped</u>

now $\omega_1 = \sqrt{\omega_0^2 - \beta^2} = i\,\omega_2$

where $\omega_2 \equiv \sqrt{\beta^2 - \omega_0^2}$ note $\omega_2 < \beta$

Now the solution is

$$x(t) = A e^{-\beta t} e^{-\omega_2 t} + B e^{-\beta t} e^{\omega_2 t}$$

$$= A e^{-(\beta+\omega_2)t} + B e^{-(\beta-\omega_2)t}$$

$$x(t) = e^{-\beta t}\left(A e^{-\omega_2 t} + B e^{\omega_2 t}\right)$$

initial conditions determine A and B

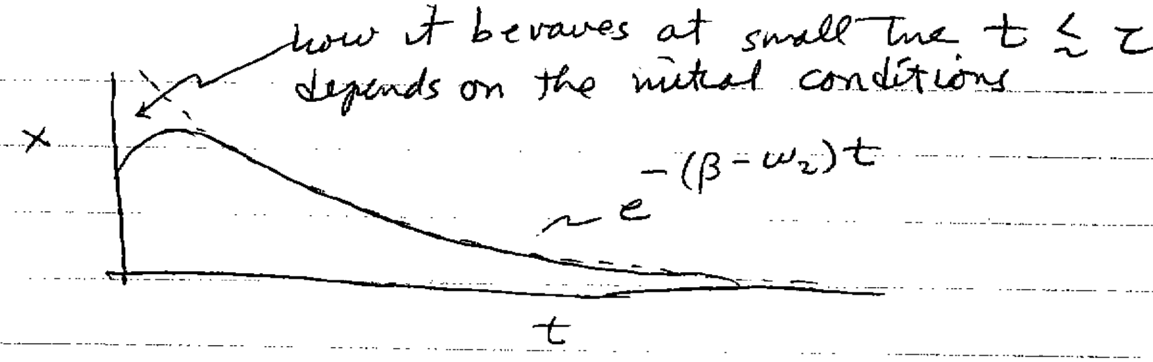
① $x(0) = A + B$

velocity $\dot{x}(t) = -(\beta+\omega_2) A e^{-(\beta+\omega_2)t} - (\beta-\omega_2)\ e^{-(\beta-\omega_1)t}$

② $\dot{x}(0) = -(\beta+\omega_2)A - (\beta-\omega_2)B$

In the general solution, the most slowly decaying term is $\sim e^{-(\beta-\omega_2)t}$

$\Rightarrow$ decay time is $\tau \sim \frac{1}{\beta-\omega_2}$ longer than under damped case, but now there are <u>no</u> oscillations



③ $\beta = \omega_0$ critical damping

Now $\omega_1 = \sqrt{\omega_0^2 - \beta^2} = 0$

solution is $x(t) = A e^{-\beta t}$
but this solution has only one free parameter

we need a second solution since there are two free initial conditions.

The second solution is (see Appendix C of text)

$x(t) = B t e^{-\beta t}$

check: $\dot{x} = B e^{-\beta t} - \beta B t e^{-\beta t}$
$\ddot{x} = -\beta B e^{-\beta t} - \beta B e^{-\beta t} + \beta^2 B t e^{-\beta t}$

$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x$

$= (-2\beta B + \beta^2 B t + 2\beta B - 2\beta^2 B t + \omega_0^2 B t) e^{-\beta t}$

$= (\beta^2 - 2\beta^2 + \omega_0^2) B t e^{-\beta t} = 0$ since $\omega_0^2 = \beta^2$

So the general solution is

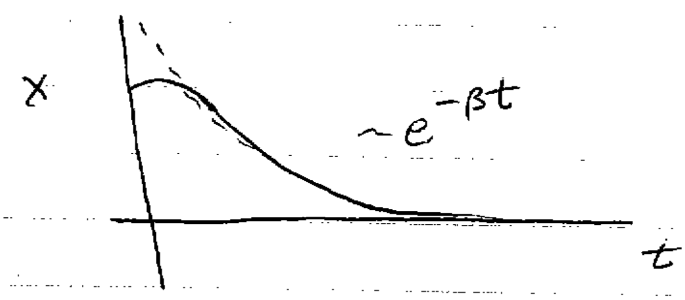
$$x(t) = (A + Bt)\, e^{-\beta t}$$

initial condition:

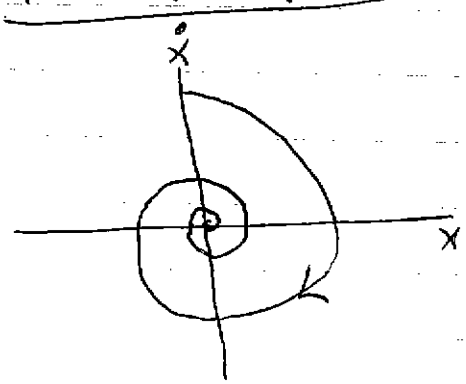
① $x(0) = A$

velocity $\quad \dot{x}(t) = -\beta (A + Bt) e^{-\beta t} + B e^{-\beta t}$

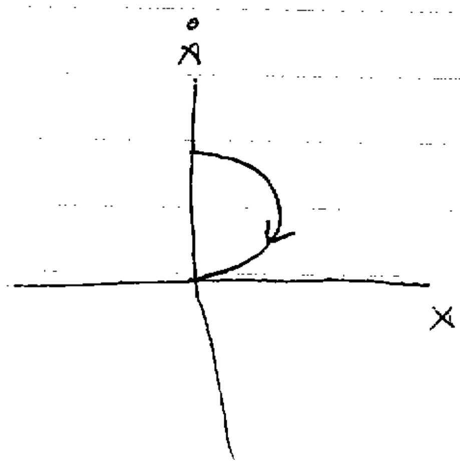
② $\dot{x}(0) = -\beta A + B$



phase space plots



underdamped



over damped