Forced oscillations

We now consider an oscillator driven by an external force

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \quad \text{external force}$$

We say that

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

is a homogeneous differential equ - each term is some derivative of x, ie $\frac{d^2x}{dt^2}$, $\frac{dx}{dt}$, $x = \frac{d^0x}{dt^0}$

While we say that

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

is an inhomogeneous differential equ - the terms in x and its derivatives are equal to some specified function that is independent of $x(t)$

One can show that the general solution to an inhomogeneous diff equ is just the sum of any particular solution plus the general solution of the corresponding homogeneous diff equ.

If $x_c(t)$ is the general solution of homog eq

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

And $x_p(t)$ is any particular solution of inhomog eq

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

($x_c(t)$ is also called the complementary solution)

Then the general solution of
$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

is given by $x_p(t) + x_c(t) = x(t)$

Recall, $x_c(t)$ must involve two free parameters and it is these free parameters we use to get the above solution $x(t)$ corresponding to any specified initial conditions $x(0)$, $\dot{x}(0)$

For the harmonic oscillator in the underdamped case we had
$$x_c(t) = A e^{-\beta t} \cos(\omega_1 t - \phi)$$
where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

For the overdamped case

$$x_c(t) = A e^{-(\beta + \omega_2)t} + B e^{-(\beta - \omega_2)t}$$
where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$

example: Suppose $F(t)$ is a constant force F_0

then a particular solution of the forced oscillator $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_0/m$ is the constant solution

$$x_p = \frac{F_0}{\omega_0^2 m} \quad \text{indep of time}$$

The general solution is therefore

$$x(t) = x_p(t) + x_c(t) \qquad \text{for underdamped case}$$

$$= \frac{F_0}{\omega_0^2 m} + A e^{-\beta t} \cos(\omega_1 t - \phi)$$

Initial conditions:

$$x(0) = \frac{F_0}{\omega_0^2 m} + A \cos$$

$$\dot{x}(0) = -\beta A \cos\phi + \omega_1 A \sin\phi \quad (\text{as in homog case})$$

Solve for A and ϕ given $x(0)$ and $\dot{x}(0)$.

Note, if we just transformed variables to $x' = x - \frac{F_0}{\omega_0^2 m}$

Then the diff eqn for x' is just

$$\ddot{x}' + 2\beta \dot{x}' + \omega_0^2 x' = 0 \; !$$

So the harmonic oscillator with const driving force is really the same as the homogeneous oscillator (no driving force) except the system oscillates about the point

$$x' = 0 \Rightarrow x = \frac{F_0}{\omega_0^2 m}$$

rather than oscillating about the point $x = 0$.

Harmonic driving force

$F(t) = F_0 \cos\omega t$ $\qquad\qquad \omega$ is any freq

$$\Rightarrow \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

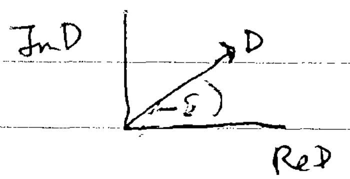
Guess the particular solution:

since drive oscillates with ω, guess that $x(t)$ also oscillates with ω.

guess:

$$x(t) = \mathrm{Re}\left[D e^{i\omega t} \right]$$

where D may be a complex number



$$D = |D| e^{-i\delta} \qquad |D| = \sqrt{(\mathrm{Re}D)^2 + (\mathrm{Im}D)^2}$$

$$\tan\delta = -\mathrm{Im}D/\mathrm{Re}D$$

$$x(t) = \mathrm{Re}\left[|D| e^{i(\omega t - \delta)} \right] = |D| \cos(\omega t - \delta)$$

similarly we can write

$$F(t) = \mathrm{Re}\left[\frac{F_0}{m} e^{i\omega t} \right]$$

We use the complex forms in the diff equations, solve, and take the real part of the complex solution at the end.

For $x(t) = De^{i\omega t}$

$\dot{x}(t) = i\omega D e^{i\omega t}$

$\ddot{x}(t) = -\omega^2 D e^{i\omega t}$

substitute into diff equ to get

$$\left[-\omega^2 D + 2i\beta\omega D + \omega_0^2 D\right] e^{i\omega t} = \frac{F_0}{m} e^{i\omega t}$$

$$\left[-\omega^2 + 2i\beta\omega + \omega_0^2\right] D = \frac{F_0}{m}$$

$$D = \frac{F_0/m}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

note D is a complex number!

We can write D in terms of its real and imaginary parts as follows

$$D = \frac{F_0/m}{(\omega_0^2 - \omega^2 + 2i\beta\omega)} \frac{(\omega_0^2 - \omega^2 - 2i\beta\omega)}{(\omega_0^2 - \omega^2 - 2i\beta\omega)}$$

$$= \frac{(F_0/m)(\omega_0^2 - \omega^2 - 2i\beta\omega)}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

Now amplitude of solution is

$$|D| = \sqrt{(\mathrm{Re}\, D)^2 + (\mathrm{Im}\, D)^2}$$

$$|D| = \frac{F_0/m}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

$$\boxed{|D| = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}}$$

and the phase of D is given by

$$\tan\delta = \frac{-\operatorname{Im} D}{\operatorname{Re} D} = \boxed{\frac{+2\beta\omega}{\omega_0^2-\omega^2} = \tan\delta}$$

So particular solution is

$$x_p(t) = \operatorname{Re}\left[D e^{i\omega t}\right]$$

$$= \operatorname{Re}\left[|D| e^{i(\omega t-\delta)}\right]$$

$$= |D|\cos(\omega t-\delta)$$

with $|D|$ and δ as above.

Note: the solution in general oscillates with a finite phase shift δ with respect to the drive.

The general solution to the harmonically driven oscillator is now just

$$x(t) = x_p(t) + x_c(t)$$

Note that since the complementary solution

$$x_c(t) = A e^{-\beta t} \cos(\omega_1 t - \phi)$$

(solve for A and δ in terms of initial conditions $x(0)$ and $\dot{x}(0)$)

always decays as $e^{-\beta t}$, then after a sufficiently long time, $t \gg 1/\beta$, the contribution to the total solution from $x_c(t)$ becomes negligible, and only the pure oscillation $x_p(t)$ remains. Because of this, $x_c(t)$ is usually called the transient part of the solution.

Resonance

Focus now on $x_p(t)$, the only piece that remains at long time, after the transient has decayed away

$$x_p(t) = |D| \cos(\omega t - \delta)$$

$$|D| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

Consider the behavior of $|D|$ and δ as functions of the freq of the driving force ω

$|D|$ has its maximum when $\frac{d|D|}{d\omega} = 0$

$$\Rightarrow \quad -\frac{1}{2}\frac{2(\omega_0^2-\omega^2)(-2\omega)+8\beta^2\omega}{[(\omega_0^2-\omega^2)^2-4\beta^2\omega^2]^{3/2}} = 0$$

$$\Rightarrow \quad 8\beta^2\omega - 4\omega(\omega_0^2-\omega^2) = 0$$

$$2\beta^2 - (\omega_0^2-\omega^2) = 0$$

$$\omega^2 = \omega_0^2 - 2\beta^2$$

Resonance freq ω_R $\Rightarrow$ $\omega_R = \sqrt{\omega_0^2 - 2\beta^2} \approx \omega_0$ when $\beta \ll \omega_0$

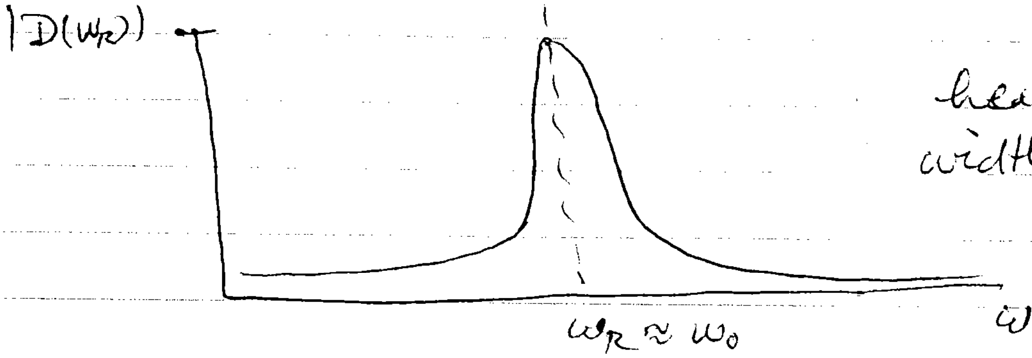
$$|D(\omega_R)| = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega_R^2)^2+4\beta^2\omega_R^2}}$$

$$= \frac{F_0/m}{\sqrt{(\omega_0^2-\omega_0^2+2\beta^2)^2+4\beta^2\omega_R^2}}$$

$$= \frac{F_0/m}{\sqrt{4\beta^4+4\beta^2\omega_R^2}}$$

$$= \frac{F_0/m}{2\beta\sqrt{\beta^2+\omega_R^2}}$$

$$|D(\omega_R)| = \frac{F_0/m}{2\beta\sqrt{\omega_0^2-\beta^2}}$$



height increases and width decreases as $\beta \to 0$

Note: as β get smaller, the amplitude of the responce at resonance ω_R gets larger. As $\beta \to 0$, $|D(\omega_R)| \to \infty$.

One defines "Quality factor"

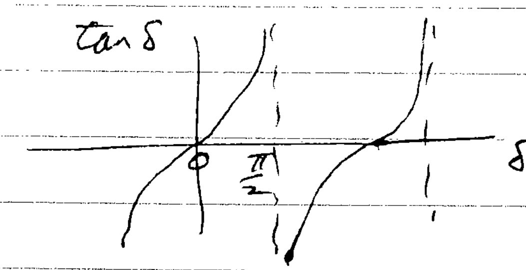
$$Q = \frac{\omega_0}{2\beta}$$

(for quartz oscillator $Q \sim 10^4$
atomic oscillators $Q \sim 10^8$

a measure of how sharp the resonance is

Now consider the phase δ.

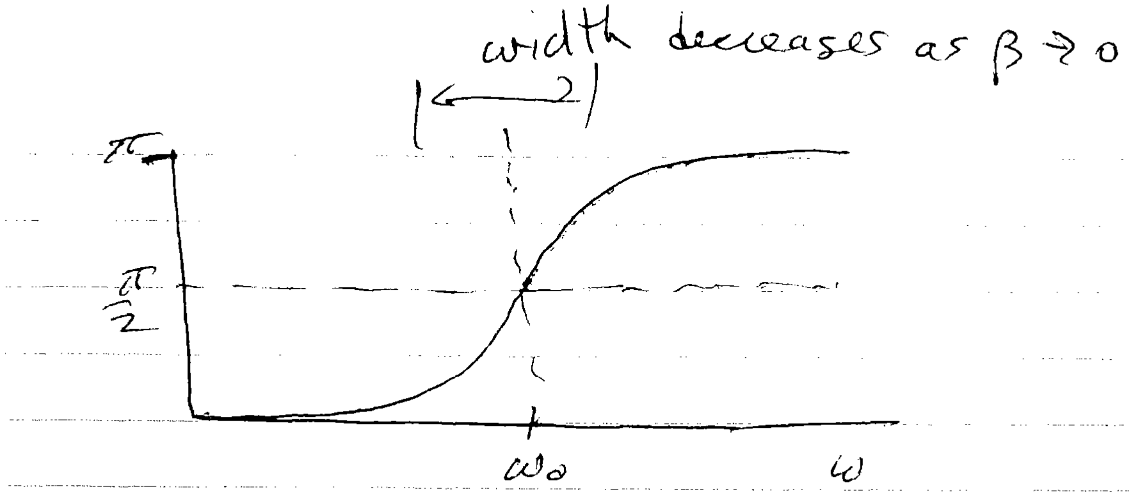
$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$



when $\omega < \omega_0$, $\tan\delta > 0$ $\to +\infty$ as $\omega \to \omega_0$
$\omega_0 < \omega$, $\tan\delta < 0$ $\to -\infty$ as $\omega_0 \leftarrow \omega$

so when $\omega = \omega_0$, $\delta = \frac{\pi}{2}$

when $\omega = 0$, $\tan\delta = 0$ $\Rightarrow \delta = 0$
when $\omega \to \infty$, $\tan\delta \sim -\frac{1}{\omega} \to -0$, $\Rightarrow \delta = \pi$



For $\omega \ll \omega_0$, δ is small

$\Rightarrow$ responce is more or less in phase with drive

For $\omega \gg \omega_0$, $\delta \sim \pi$

$\Rightarrow$ responce is ~~exactly out~~ π of phase with drive

$$x_p(t) = |D| \cos(\omega t - \pi) = -|D| \cos(\omega t)$$

For $\omega = \omega_0$ $\delta = \pi/2$

$\Rightarrow$ responce is $\pi/2$ out of phase with drive

$$x_p(t) = |D| \cos(\omega t - \tfrac{\pi}{2}) = |D| \sin(\omega t)$$

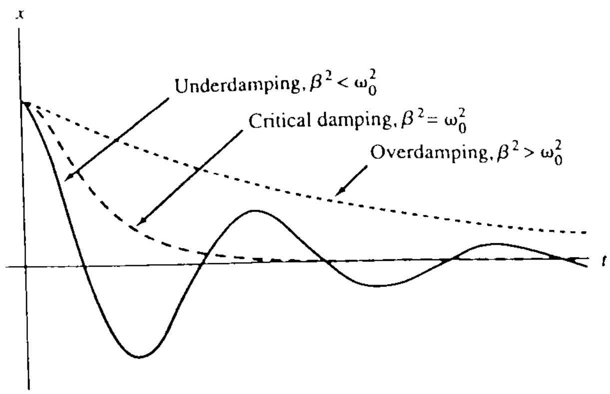


FIGURE 3-5

decay of amplitude

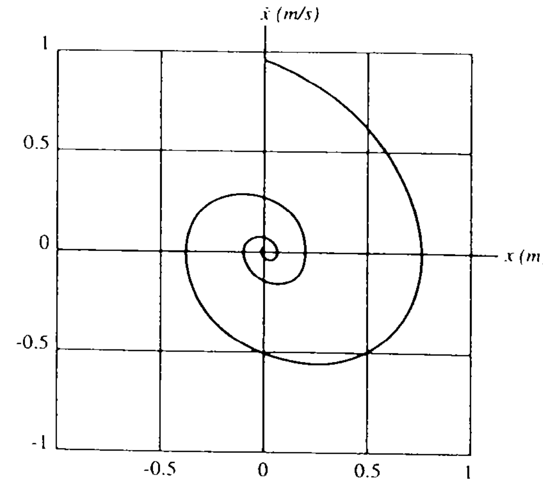
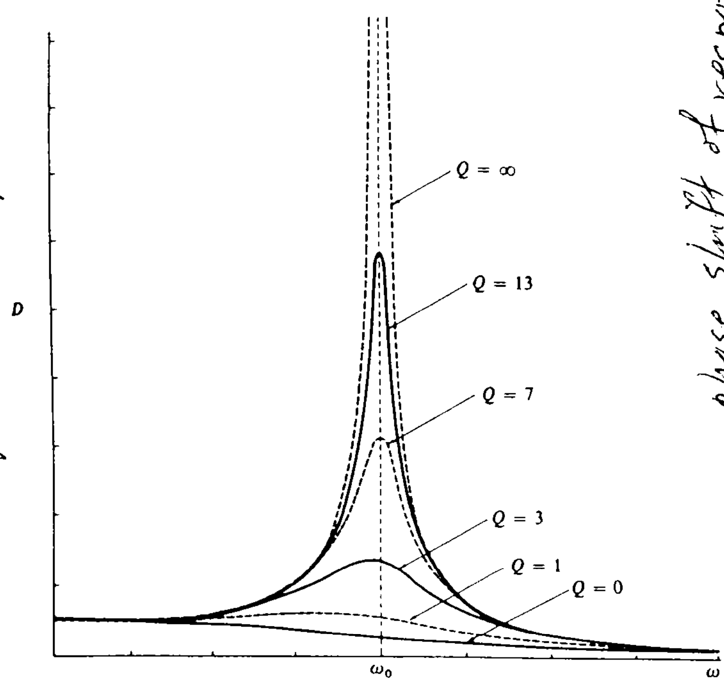


FIGURE 3-9b

phase space plot

free oscillator

amplitude response



phase shift of response

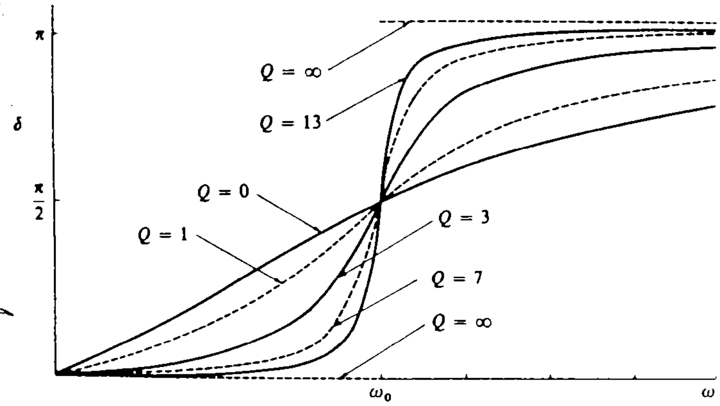


FIGURE 3-15

driven oscillator