Energy + Work in a driven oscillator

$$x_p(t) = |D| \cos(\omega t - \delta)$$
$$\dot{x}_p(t) = -\omega |D| \sin(\omega t - \delta)$$

$$F(t) = F_0 \cos(\omega t)$$

The work done on the oscillator by the driving force, in one period of oscillation T is

$$W = \int F\, dx = \int_0^T F \frac{dx}{dt}\, dt$$
$$= -\int_0^T dt\ F_0 \cos(\omega t)\, \omega |D| \sin(\omega t - \delta)$$
$$= -\omega |D| F_0 \int_0^T dt\ \cos(\omega t)\left[\sin\omega t \cos\delta - \cos\omega t \times \sin\delta\right]$$
$$= -\omega |D| F_0 \cos\delta \int_0^T dt \cos\omega t \sin\omega t$$
$$+ \omega |D| F_0 \sin\delta \int_0^T dt \cos^2\omega t$$

Now $\int_0^T dt \cos\omega t \sin\omega t = \int_0^T dt \frac{\sin(2\omega t)}{2} = 0$

$$\int_0^T dt \cos^2\omega t = \frac{1}{2} T$$

So

$$W = \frac{\omega |D| F_0}{2} \sin\delta\ T$$ = energy absorbed by oscillator from external force in one

average power absorbed in one period of oscillation is

$$P = \frac{W}{T} = \frac{\omega |D| F_0 \sin\delta}{2}$$

Recall, $D = |D| e^{-i\delta}$

so $|D| \sin\delta = -\operatorname{Im} D$

So power P is proportional to that part of the responce that is $\frac{\pi}{2}$ out of phase with the drive

Earlier we fand

$$-\operatorname{Im} D = \frac{(F_0/m)\, 2\beta\omega}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

So
$$\boxed{P = \frac{F_0^2}{2m} \frac{2\beta\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

when $\omega \sim \omega_0 \gg \beta$ then

$$\omega_0^2 - \omega^2 = (\omega_0+\omega)(\omega_0-\omega) \simeq 2\omega_0(\omega_0-\omega)$$

then

$$P \sim \frac{F_0^2}{2m} \frac{2\beta\omega_0^2}{4\omega_0^2(\omega_0-\omega)^2+4\beta^2\omega_0^2}$$

$$\sim \frac{F_0^2}{8m} \frac{2\beta}{(\omega_0-\omega)^2+\beta^2}$$

this is called the Lorentzian or Briet-Wigner shape

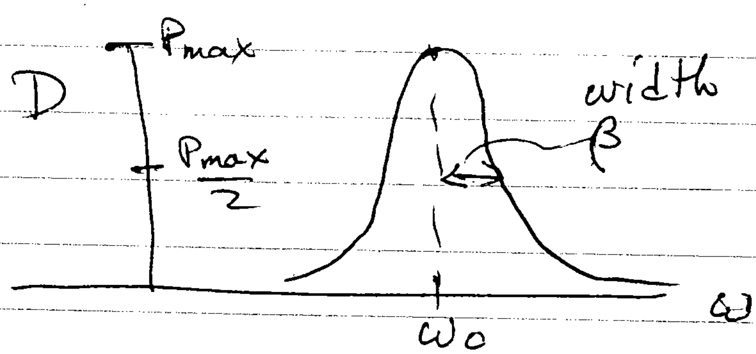
P has max at $\omega = \omega_0$ $\Leftarrow$ resonance in absorbed energy is at $\omega = \omega_0$, not ω_R

$$P_{max} = \frac{F_0^2}{8m} \frac{2}{\beta}$$

when $|\omega - \omega_0| = \beta$ then

$$P = \frac{F_0^2}{8m} \frac{2\beta}{2\beta^2} = \frac{F_0^2}{8m} \frac{1}{\beta} = \frac{1}{2} P_{max}$$

therefore the damping β is also the half width at half height of the power resonance



Lorentzian has narrower peak, but wider tails than Gaussian $e^{-\frac{1}{2}(\omega-\omega_0)^2/\sigma^2}$

Prob 3-18

Energy stored in oscillator

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m \omega^2 |D|^2 \sin^2(\omega t - \delta) + \frac{1}{2} k |D|^2 \cos^2(\omega t - \delta)$$

If one averages over one period of oscillation

$$\langle E \rangle = \frac{1}{T} \int_0^T dt\, E(t)$$

$$= \frac{1}{2} m \omega^2 |D|^2 \left(\frac{1}{2}\right) + \frac{1}{2} k |D|^2 \left(\frac{1}{2}\right)$$

$$\langle E \rangle = \frac{1}{4} |D|^2 (m\omega^2 + k)$$

use $m\omega_0^2 = k$

$$\langle E \rangle = \frac{1}{4} |D|^2 k \left(\left(\frac{\omega}{\omega_0}\right)^2 + 1 \right)$$

$$\boxed{\langle E \rangle = \frac{k}{4} |D|^2 \left(\frac{\omega^2 + \omega_0^2}{\omega_0^2} \right)}$$

use $|D|^2 = \dfrac{(F_0/m)^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$

$$\boxed{\langle E \rangle = \frac{k}{4} \left(\frac{F_0}{m}\right)^2 \frac{1}{\omega_0^2} \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

at $\omega = \omega_0$

$$\langle E \rangle_0 = \frac{k}{4}\left(\frac{F_0}{m}\right)^2 \frac{2\omega_0^2}{\omega_0^2} \frac{1}{4\beta^2\omega_0^2}$$

$$= \frac{k}{8}\left(\frac{F_0}{m}\right)^2 \frac{1}{\beta^2\omega_0^2}$$

use $k = m\omega_0^2$

$$\boxed{\langle E \rangle_0 = \frac{1}{8}\frac{F_0^2}{m}\frac{1}{\beta^2} \quad \text{at resonance}}$$

Energy absorbed in one oscillation at resonance is

$$W_0 = P_{max} T = \frac{F_0^2}{8m}\frac{2}{\beta} \cdot \left(\frac{2\pi}{\omega_0}\right)$$

$$= 2\pi \frac{F_0^2}{8m}\frac{2}{\beta\omega_0}$$

$$\frac{\text{energy absorbed}}{\text{energy stored}} = \frac{W_0}{\langle E \rangle_0} = \frac{2\pi \frac{F_0^2}{8m}\frac{2}{\beta\omega_0}}{\frac{F_0^2}{8m}\frac{1}{\beta^2}} = 2\pi\left(\frac{2\beta}{\omega_0}\right)$$

$$\frac{\text{energy absorbed}}{\text{energy stored}} = 2\pi Q^{-1} \qquad Q = \frac{\omega_0}{2\beta}$$

quality factor