Linearity and Superposition

We can write the differential operator that appears in the driven harmonic oscillator problem as

$$L[x] \equiv \ddot{x} + 2\beta\dot{x} + \omega_0^2 x$$

$L[x]$ is a linear operator

$$\Rightarrow L[x_1 + x_2] = L[x_1] + L[x_2]$$

$$\frac{d^2}{dt^2}(x_1+x_2) = \frac{d^2x_1}{dt^2} + \frac{dx_2^2}{dt^2} = \ddot{x}_1 + \ddot{x}_2 \quad \text{etc.}$$

If x_1 is a particular solution to

$$L[x] = F_1(t) \quad , \text{ ie } \quad L[x_1] = F_1(t)$$

and x_2 is a particular solution to

$$L[x] = F_2(t) \quad , \text{ ie } \quad L[x_2] = F_2(t)$$

then $x_1 + x_2$ is a particular solution to

$$L[x] = F_1(t) + F_2(t)$$

Since $L[x_1+x_2] = L[x_1] + L[x_2] = F_1 + F_2$

In general, if x_n is a particular solution to

$L[x] = F_n(t)$

then $x = \sum_n \alpha_n x_n + x_c$ is the general solution

to

$L[x] = \sum_n \alpha_n F_n(t)$, where

x_c is the general solution to $L[x] = 0$.

For $F_n(t) = \mathrm{Re}\left[F_n e^{i\omega_n t}\right]$ where $F_n = |F_n| e^{-i\phi_n}$ may be complex

then a particular solution to $L[x] = \frac{F_n(t)}{m}$ is

$$x_n(t) = \mathrm{Re}\left[D_n e^{i\omega_n t}\right] \quad \text{where } D_n = \frac{F_n/m}{\omega_0^2 - \omega_n^2 + 2i\beta\omega_n}$$

$\Rightarrow$ for $F_n = |F_n| \cos(\omega_n t - \phi_n)$, a solution is

$$x_n(t) = |D_n| \cos(\omega_n t - \phi_n - \delta_n)$$

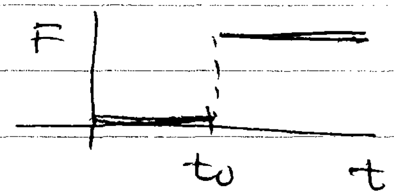
where $$|D_n| = \frac{|F_n|/m}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\beta^2\omega_n^2}}$$

$$\tan \delta_n = \frac{2\beta\omega_n}{\omega_0^2 - \omega_n^2}$$

This is important because <u>any</u> periodic $F(t)$ can always be written as a sum of harmonic functions

Response to a step function

$$F(t) = \begin{cases} 0 & t < t_0 \\ F_0 & t > t_0 \end{cases}$$



a particular solution to $L[x] = \frac{F(t)}{m}$ is

$$x_p(t) = 0 \qquad t < t_0$$
$$x_p(t) = \frac{F_0}{\omega_0^2 m} \qquad t > t_0$$

general solution is:

$t < t_0$:
$$x_1(t) = x_p(t) + x_c(t)$$
$$= 0 + e^{-\beta t}(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$

$t > t_0$:
$$x_2(t) = x_p(t) + x_c(t)$$
$$= \frac{F_0}{\omega_0^2 m} + e^{-\beta t}(A_2 \cos \omega_1 t + B_2 \sin \omega_1 t)$$

Assume initial condition $x_1(0) = \dot{x}_1(0) = 0$

$x_1(0) = 0 \Rightarrow A_1 = 0$, $\dot{x}_1 = -\beta e^{-\beta t}(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t) + e^{-\beta t}(-\omega_1 A_1 \sin \omega_1 t + \omega_1 B_1 \cos \omega_1 t)$

$\dot{x}_1(0) = 0 \Rightarrow -\beta A_1 + \omega_1 B_1 = 0 \Rightarrow B_1 = 0$

$$\Rightarrow \quad \frac{e^{-\beta t_0}\,\omega_1}{\beta\cos\omega_1 t_0 + \omega_1 \sin\omega_1 t_0}\, B_2 = \frac{-F_0}{\omega_0^2 m}$$

$$\Rightarrow \begin{cases} B_2 = \dfrac{-F_0}{\omega_0^2 m}\dfrac{e^{\beta t_0}}{\omega_1}(\beta\cos\omega_1 t_0 + \omega_1\sin\omega_1 t_0) \\ A_2 = \dfrac{-F_0}{\omega_0^2 m}\dfrac{e^{\beta t_0}}{\omega_1}(-\beta\sin\omega_1 t_0 + \omega_1\cos\omega_1 t_0) \end{cases}$$

Substitute into $x_2(t) = \dfrac{F_0}{\omega_0^2 m} + e^{-\beta t}(A_2\cos\omega_1 t + B_2\sin\omega_1 t)$

to get

$$x_2(t) = \frac{F_0}{\omega_0^2 m}\Big[1 - \frac{e^{-\beta(t-t_0)}}{\omega_1}\big(-\beta\sin\omega_1 t_0\cos\omega_1 t + \omega_1\cos\omega_1 t_0\cos\omega_1 t$$
$$+\beta\cos\omega_1 t_0\sin\omega_1 t + \omega_1\sin\omega_1 t_0\sin\omega_1 t$$

using trig formular for cos (A+B) and sin (A+B)

$$\boxed{x_2(t) = \frac{F_0}{\omega_0^2 m}\Big[1 - \frac{e^{-\beta(t-t_0)}}{\omega_1}\big(\omega_1\cos\omega_1(t-t_0) + \beta\sin\omega_1(t-t_0)\big)\Big]}$$

$$x_2(t) = \frac{F_0}{\omega_0^2 m}\Big[1 - e^{-\beta(t-t_0)}\big(\cos\omega_1(t-t_0) + \frac{\beta}{\omega_1}\sin\omega_1(t-t_0)\big)\Big]$$

note: we could have gotten this result in a smarter and simpler way as follows:

Define $t' = t - t_0$

Solution to $t > t_0$, or $t' > 0$, could have be written as

$$x_2(t') = \frac{F_0}{\omega_0^2 m} + e^{-\beta t'}\left[A\cos(\omega_1 t') + B\sin(\omega_1 t')\right]$$

initial conditions are $x(t_0) = x(t'=0) = 0$

$\dot{x}(t_0) = \dot{x}'(t'=0) = 0$

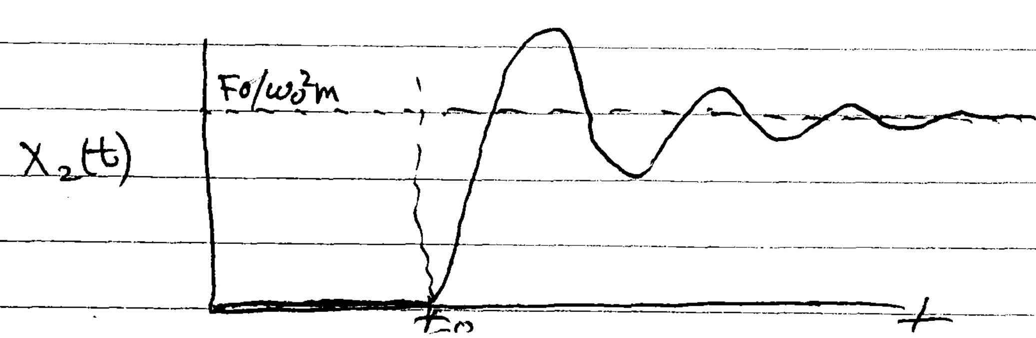
$x_2(t'=0) = 0 \Rightarrow \frac{F_0}{\omega_0^2 m} + A = 0 \Rightarrow A = -\frac{F_0}{\omega_0^2 m}$

$\dot{x}_2(t'=0) = 0 \Rightarrow -\beta A + \omega_1 B = 0 \Rightarrow B = \frac{\beta A}{\omega_1} = \frac{-F_0}{\omega_0^2 m}\frac{\beta}{\omega_1}$

$$\Rightarrow x_2(t') = \frac{F_0}{\omega_0^2 m}\left[1 - e^{-\beta t'}\left(\cos\omega_1 t' + \frac{\beta}{\omega_1}\sin\omega_1 t'\right)\right]$$

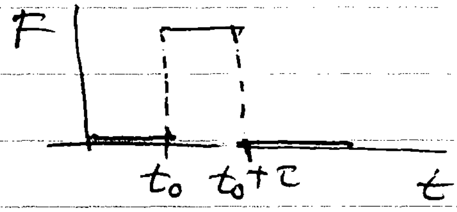
$$x_2(t) = \frac{F_0}{\omega_0^2 m}\left[1 - e^{-\beta(t-t_0)}\left(\cos\omega_1(t-t_0) + \frac{\beta}{\omega_1}\sin\omega_1(t-t_0)\right)\right]$$

$\Rightarrow$ same result as before

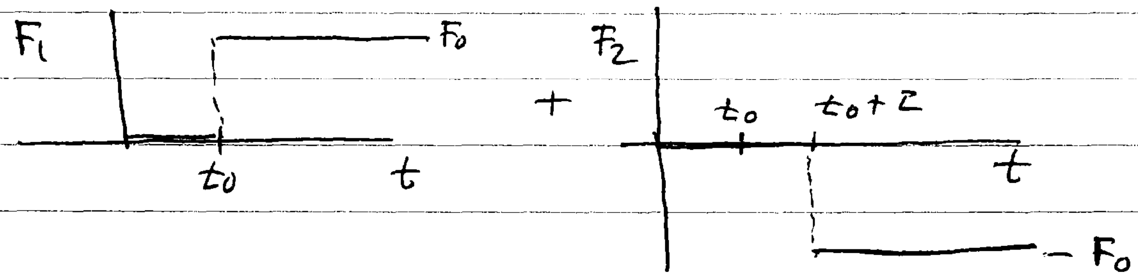


Response to an impulse

$$F(t) = \begin{cases} 0 & t < t_0 \\ F_0 & t_0 < t < t_0 + \tau \\ 0 & t_0 + \tau < t \end{cases}$$



Regard F(t) as superposition of



By linearity, we know that solution to $F_1 + F_2$ is just solution to F_1, plus solution to F_2

$$x_1(t) = \begin{cases} 0 & t < t_0 \\ \frac{F_0}{\omega_0^2 m}\left[1 - e^{-\beta(t-t_0)}\left(\cos\omega_1(t-t_0) + \frac{\beta}{\omega_1}\sin\omega_1(t-t_0)\right)\right] & t > t_0 \end{cases}$$

$$x_2(t) = \begin{cases} 0 & t < t_0+\tau \\ \frac{-F_0}{\omega_0^2 m}\left[1 - e^{-\beta(t-t_0-\tau)}\left(\cos\omega_1(t-t_0-\tau) + \frac{\beta}{\omega_1}\sin\omega_1(t-t_0-\tau)\right)\right] & t > t_0+\tau \end{cases}$$

Solution to $F = F_1 + F_2$ is $x = x_1 + x_2$

Note $x(t)$ has correct initial conditions $x(0) = \dot{x}(0) = 0$ and ~~is continuous~~ $x(t)$ and $\dot{x}(t)$ are continuous at $t = t_0$ and $t = t_0 + \tau$. So $x(t)$ is solution to our problem

Suppose now that τ is very small.

We can write to first order in τ

$$e^{-\beta(t-t_0-\tau)} = e^{-\beta(t-t_0)} e^{\beta\tau} = e^{-\beta(t-t_0)} (1+\beta\tau)$$

$$\cos\omega_1(t-t_0-\tau) = \cos\omega_1(t-t_0)\cos\omega_1\tau + \sin\omega_1(t-t_0)\sin\omega_1\tau$$
$$\cos\omega_1(t-t_0) + \omega_1\tau\sin\omega_1(t-t_0)$$

$$\sin\omega_1(t-t_0-\tau) = \sin\omega_1(t-t_0)\cos\omega_1\tau - \cos\omega_1(t-t_0)\sin\omega_1\tau$$
$$\simeq \sin\omega_1(t-t_0) - \omega_1\tau\cos\omega_1(t-t_0)$$

So, to $o(\tau)$

$$X_2(t) \cong \frac{-F_0}{\omega_0^2 m}\Big[1 - e^{-\beta(t-t_0)}\Big[\Big(\cos\omega_1(t-t_0) + \frac{\beta}{\omega_1}\sin\omega_1(t-t_0)\Big)$$
$$+ \beta\tau\Big(\cos\omega_1(t-t_0) + \frac{\beta}{\omega_1}\sin\omega_1(t-t_0)\Big)$$
$$+ \omega_1\tau\Big(\sin\omega_1(t-t_0) - \frac{\beta}{\omega_1}\cos\omega_1(t-t_0)\Big)\Big]\Big]$$

For $t > t_0+\tau$,

$$X(t) = X_1 + X_2$$

when add, the $O(1)$ term in X_2 exactly cancels the contribution from X_1 leaving only the $o(\tau)$ terms left

$$= \frac{+F_0}{\omega_0^2 m} e^{-\beta(t-t_0)}\Big[+\beta\tau\cos\omega_1(t-t_0) + \frac{\beta^2\tau}{\omega_1}\sin\omega_1(t-t_0)$$
$$+ \omega_1\tau\sin\omega_1(t-t_0) - \beta\tau\cos\omega_1(t-t_0)\Big]$$

$$X(t) = \frac{F_0}{\omega_0^2 m}\Big(\omega_1\tau + \frac{\beta^2\tau}{\omega_1}\Big) e^{-\beta(t-t_0)} \sin\omega_1(t-t_0) \quad \text{for } t > t_0+\tau$$

Now as $\tau \to 0$, the region $t_0 < t < t+\tau$ becomes vanishingly small. So the solution becomes

$$x(t) = \begin{cases} 0 & t < t_0 \\ \dfrac{F_0 \tau}{\omega_0^2 m}\left(\omega_1 + \dfrac{\beta^2}{\omega_1}\right) e^{-\beta(t-t_0)} \sin \omega_1 (t-t_0) & t > t_0 \end{cases}$$

For this solution to make sense, we need that $F_0 \tau \to$ constant as $\tau \to 0$, ie F_0 diverges as $\tau \to 0$ in such a way that $F_0 \tau$ stays constant

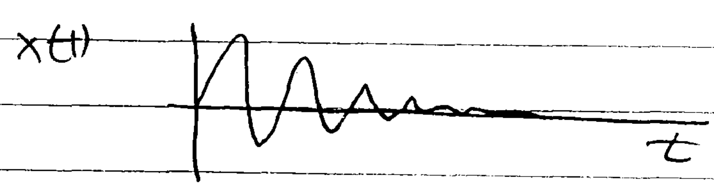
Note: $F_0 \tau = \int_{-\infty}^{\infty} F(t)\, dt = \int \frac{dp}{dt}\, dt = p_{final} - p_{initial}$

gives change in momentum of particle as result of the impulse force, $F(t)$.

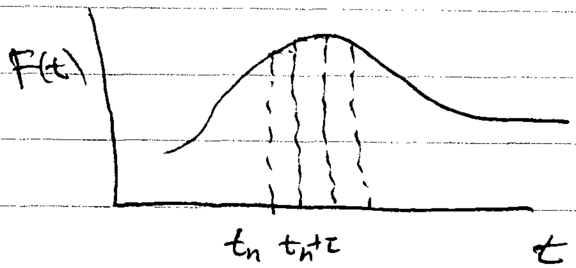
Note, using $\omega_1^2 = \omega_0^2 - \beta^2$, we can write

$$\frac{1}{\omega_0^2}\left(\omega_1 + \frac{\beta^2}{\omega_1}\right) = \frac{1}{\omega_0^2}\left(\frac{\omega_1^2 + \beta^2}{\omega_1}\right) = \frac{1}{\omega_0^2}\,\frac{\omega_0^2}{\omega_1} = \frac{1}{\omega_1}$$

So
$$\boxed{x(t) = \begin{cases} 0 & t < t_0 \\ \dfrac{F_0 \tau}{m\omega_1} e^{-\beta(t-t_0)} \sin \omega_1 (t-t_0) & t > t_0 \end{cases}}$$



Now, any general force $F(t)$ can always be written as a sum of impulse forces



We get the solution for a particle initially at rest.

$$F(t) = \sum_n F_n(t) \quad \text{where } F_n(t) = \begin{cases} 0 & t < t_n \\ F(t_n) & t_n < t < t_n+\tau \\ 0 & t_n + \tau < t \end{cases}$$

Solution $x_n(t)$ for $F_n(t)$ alone is as $\tau \to 0$

$$x_n(t) = \begin{cases} 0 & t < t_n \\ \dfrac{F(t_n)\tau}{m\omega_1} e^{-\beta(t-t_n)} \sin \omega_1 (t-t_n) & t > t_n \end{cases}$$

Solution for F is then

$$x(t) = \sum_n x_n(t) = \sum_{t_n < t} \frac{F(t_n)\tau}{m\omega_1} e^{-\beta(t-t_n)} \sin \omega_1 (t-t_n)$$

as $\tau \to 0$

$$\boxed{x(t) = \int_{-\infty}^{t} dt' \, \frac{F(t')}{m\omega_1} e^{-\beta(t-t')} \sin \omega (t-t')}$$

has the form $x(t) = \int_{-\infty}^{t} dt' F(t') \, G(t-t')$

where $G(t-t') = \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin \omega(t-t')$ is the Green's function for the harmonic oscillator, Can derive the same result using

Fourier transforms

$$F(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, F_\omega \quad , \quad F_\omega = \int_{-\infty}^{\infty} dt\, e^{-i\omega t}\, F(t)$$

$$\Rightarrow x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, D_\omega$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, \frac{F_\omega}{m}\, \frac{1}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{e^{i\omega t}}{m}\, \frac{1}{\omega_0^2 - \omega^2 + 2i\beta\omega} \int_{-\infty}^{\infty} dt'\, e^{-i\omega t'}\, F(t')$$

$$= \int_{-\infty}^{\infty} dt'\, \frac{F(t')}{m} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t'-t)}\, \frac{1}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

can do ω integration by complex contour integration to get

$$\frac{e^{-\beta(t-t')}}{\omega_1} \sin\omega(t-t')$$

we recover earlier result