between the electric field of the EM wave and the resulting current due to the conduction electrons

For a single harmonic electromagnetic wave we can write for the fields:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] \quad \text{electric field}$$
$$\vec{H}(\vec{r},t) = \mathrm{Re}\left[\vec{H}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] \quad \text{magnetic field}$$

The relation between the amplitudes $\vec{E}_0$ and $\vec{H}_0$ and between the wavevector $\vec{k}$ and frequency ω are determined by Maxwell's Equations.

We will look for solutions for a transverse propagating wave, ie with $\vec{E}_0 \perp \vec{k}$.

Macroscopic Maxwell's Equations (in CGS units)

Gauss $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$ Gauss

Ampere $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ Faraday

We will ignore magnetization effects, ie $\mu = 0$ and $\vec{B} = \vec{H}$

We will ignore polarization from bound electrons, ie $\varepsilon = 1$ and $\vec{E} = \vec{D}$

$\vec{j}$ is current due to conduction electrons

ρ is any locally non-neutral charge density due to variations in conduction electron density

Above become

$$\vec\nabla\cdot\vec E = 4\pi\rho \qquad\qquad \vec\nabla\cdot\vec H = 0$$

$$\vec\nabla\times\vec H = \frac{4\pi}{c}\vec j + \frac{1}{c}\frac{\partial\vec E}{\partial t} \qquad\qquad \vec\nabla\times\vec E = -\frac{1}{c}\frac{\partial\vec H}{\partial t}$$

Substitute into the above the single harmonic forms for $\vec E$ and $\vec H$.

Gauss $\vec\nabla\cdot\vec E = i\vec k\cdot\vec E_0 e^{i(\vec k\cdot\vec r-\omega t)} = 0$ since assumed $\vec E_0\perp\vec k$

$\Rightarrow \rho = 0$ transverse EM wave induces no local charge density

Gauss $\vec\nabla\cdot\vec H = i\vec k\cdot\vec H_0 e^{i(k\cdot r-\omega t)} = 0$

$\Rightarrow \vec H_0\perp\vec k$ so magnetic field is also transverse

(1) Faraday $\vec\nabla\times\vec E = -\frac{1}{c}\frac{\partial\vec H}{\partial t} \Rightarrow i\vec k\times\vec E_0 = i\frac{\omega}{c}H_0$

(2) Ampere $\vec\nabla\times\vec H = \frac{4\pi}{c}\vec j + \frac{1}{c}\frac{\partial\vec E}{\partial t} \Rightarrow i\vec k\times\vec H_0 = \frac{4\pi}{c}\vec j_0 - \frac{i\omega}{c}\vec E_0$

where assumed $\vec j(\vec r,t) = \mathrm{Re}\left[\vec j_0 e^{i(\vec k\cdot\vec r-\omega t)}\right]$

Multiply (1) by $\vec k\times$

$$i\vec k\times(\vec k\times\vec E_0) = \underbrace{i(\vec k\cdot\vec E_0)\vec k}_{=0\text{ since }\vec k\perp\vec E_0} - ik^2\vec E_0 = i\frac{\omega}{c}\vec k\times\vec H_0$$

so $i\vec k\times\vec H_0 = -i\frac{k^2c}{\omega}\vec E_0$

Substitute from (2)

$$i\vec k\times\vec H_0 = -i\frac{k^2c}{\omega}\vec E_0 = \frac{4\pi}{c}\vec j_0 - \frac{i\omega}{c}\vec E_0$$

$$\Rightarrow \quad \vec{E}_0\left(k^2-\frac{\omega^2}{c^2}\right) = \frac{4\pi i\omega}{c^2}\vec{j}_0$$

For a vacuum, $\vec{j}_0 = 0 \quad \Rightarrow \quad \omega^2 = c^2k^2$

For a metal $\quad \vec{j}_0 = \sigma(\omega)\vec{E}_0 \qquad \sigma(\omega)$ is ac conductivity

$$\Rightarrow \quad \vec{E}_0\left(k^2-\frac{\omega^2}{c^2}\right) = \frac{4\pi i\,\omega}{c^2}\sigma(\omega)\vec{E}_0$$

$$\Rightarrow \quad \vec{E}_0\left(k^2-\frac{\omega^2}{c^2}-\frac{4\pi i\omega}{c^2}\sigma(\omega)\right) = 0$$

$$\Rightarrow \quad \boxed{k^2 = \frac{\omega^2}{c^2}\left(1+\frac{4\pi i\,\sigma(\omega)}{\omega}\right)} \leftarrow$$

"dispersion relation" for EM waves in a metal.

$$\sigma(\omega) = \frac{\sigma_{dc}}{1-i\omega\tau} \qquad \sigma_{dc} = \frac{ne^2\tau}{m}$$

For low frequencies, $\omega\tau \ll 1$, ie frequency much smaller than collision rate, $\sigma(\omega) \approx \sigma_{dc}$

$$\text{then} \quad k^2 = \frac{\omega^2}{c^2}\left(1+\frac{4\pi i\,\sigma_{dc}}{\omega}\right)$$

For fixed real ω, k will have a large imaginary part. When ω low enough that $4\pi\sigma_{dc}/\omega \gg 1$ then

$$k^2 = \frac{\omega^2}{c^2}\frac{4\pi i\,\sigma_{dc}}{\omega}$$ [illegible]

$$k = \frac{1}{c}\sqrt{4\pi\sigma_{dc}\,\omega}\left(\frac{1+i}{\sqrt{2}}\right)$$

k is complex number with equal real and imaginary parts.

since $\vec{E} = Re\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$

if we write $\vec{k} = \vec{k}_1 + i\vec{k}_2$

$$E = Re\left\{\vec{E}_0 e^{i(\vec{k}_1\cdot\vec{r}-\omega t)}\right] e^{-\vec{k}_2\cdot\vec{r}}$$

wave decays as it propagates into metal

In low freq limit where $k_1 \simeq k_2$, wave ~~propa~~ decays by factor 1/e for every wavelength it penetrates.

More interesting behavior in higher frequency limit, $\omega\tau \gg 1$, ie frequency large compared to collision rate.

then $\sigma(\omega) \approx \frac{\sigma_{dc}}{-i\omega\tau} = \frac{ne^2\tau}{m\omega\tau}\, i = \frac{ne^2}{m\omega}\, i$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{4\pi ne^2}{m\omega^2}\right)$$

call $\omega_p \equiv \sqrt{\frac{4\pi n e^2}{m}}$ the "plasma frequency"

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)$$ dispersion relation is <u>independent</u> of τ

In general $\omega_p \gg \frac{1}{\tau}$

For freq $\omega > \omega_p$ we have

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)$$ is positive real number

$$k = \frac{\omega}{c}\sqrt{1 - (\omega_p/\omega)^2}$$ is real

EM wave propagates through the metal with no attenuation. For $\omega > \omega_p$, the metal is <u>transparent</u> to EM waves!

But for freq $\frac{1}{\tau} \ll \omega < \omega_p$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right) < 0$$ is negative

$$k = i\frac{\omega}{c}\sqrt{\left|1 - (\omega_p/\omega)^2\right|}$$ is pure imaginary

$\vec{E}(\vec{r},t) =$ ~~E Re{E₀ e^{ikr}~~

$$\mathrm{Re}\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right]$$

$$= \mathrm{Re}\left[\vec{E}_0 e^{-i\omega t}\right] e^{-|k|\hat{n}\cdot\vec{r}}$$

field decays exponentially — waves do not propagate

$\omega < \omega_p$, EM waves get absorbed by the metal.

The crossover from absorption to transparent occurs at $\omega = \omega_p$, or at wavelength

$$\lambda_p \equiv 2\pi c/\omega_p \qquad \omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

depends only on electron density.

Compare this Drude prediction to experiment

metal	λ_p (Drude)	λ_p (expt)	(10^3 Å)
Li	1.5	2.0	
Na	2.0	2.1	
K	2.8	3.1	
Rb	3.1	3.6	
Cs	3.5	4.4	

agreement is not bad given all the simplifying approximations that we have made!

Thermal Conductivity of metals

Apply temperature gradient $\vec{\nabla} T$ across sample. A thermal current (energy current) $\vec{j}^q$ will flow.

From thermodynamics: $dE = TdS - pdV$.
Here $dV = 0$ so $dE = TdS$.
Heat $dQ = TdS$ so $dQ = dE$
$\Rightarrow$ heat current (or thermal current) = energy current

$$\vec{j}^q = -\kappa \vec{\nabla} T \qquad \text{heat equation}$$

defines <u>thermal conductivity</u> κ
heat flows from hot to cold so $\kappa \geq 0$
(thats why we define κ using a (−) sign in the heat equation)

Experiment observed the following empirical relation between κ and the dc conductivity σ

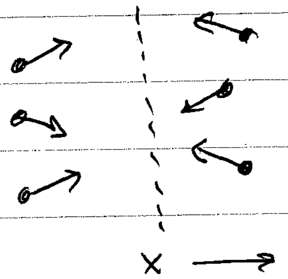
Wiedemann - Franz law

$$\frac{\kappa}{\sigma} \sim \text{const} \times T$$

where const $\simeq 2 \times 10^{-8}$ watt-ohm/$^0K^2$ is roughly the same for all metals.

Consider the energy flowing through a plane $\perp$ to $\vec{\nabla} T$

suppose $\vec{\nabla} T = \frac{dT}{dx} \hat{x}$



Electrons crossing from left to right have had their last collision on average a time τ earlier at position $x^L = x - v_x^L \tau$, where v_x^L is the average speed of these electrons in the x direction

The electrons thus have average energy $\mathcal{E}(T(x - v_x^L \tau))$ where $\mathcal{E}(T)$ is the average energy at temperature T.

Similarly the electrons crossing from right to left have had their last collision at position $x^R = x + v_x^R \tau$ and carry average energy $\mathcal{E}(T(x + v_x^R \tau))$.

The number of electrons per unit time per unit area crossing left to right is $\frac{1}{2} n v_x^L$.

The number of electrons per unit time per unit area crossing right to left is $\frac{1}{2} n v_x^R$.

(factor $\frac{1}{2}$ since half of electrons go in $+\hat{x}$ direction and $\frac{1}{2}$ go in $-\hat{x}$ direction)

$$\Rightarrow j^q = \tfrac{1}{2} m v_x^L \, \mathcal{E}\left(T(x - v_x^L \tau)\right)$$

$$- \tfrac{1}{2} m v_x^R \, \mathcal{E}\left(T(x + v_x^R \tau)\right)$$

For slow temperature variation $\frac{dT}{dx} < \frac{T_0}{\ell}$ ← ave temp, ← mean free path
we can expand

$$j^q = \tfrac{1}{2} m v_x^L \left[\mathcal{E}(T(x)) - v_x^L \tau \frac{d\mathcal{E}}{dT} \frac{dT}{dx} \right]$$

$$- \tfrac{1}{2} m v_x^R \left[\mathcal{E}(T(x)) + v_x^R \tau \frac{d\mathcal{E}}{dT} \frac{dT}{dx} \right]$$

$$= -\tfrac{1}{2} m \tau \frac{d\mathcal{E}}{dT} \frac{dT}{dx} \left[(v_x^L)^2 + (v_x^R)^2 \right]$$

$$+ \tfrac{1}{2} m \, \mathcal{E}(T(x)) \left[v_x^L - v_x^R \right]$$

<u>first term</u>: $\frac{1}{2}(v_x^L)^2 + \frac{1}{2}(v_x^R)^2 \approx \langle v_x^2 \rangle = \frac{1}{3}\langle v^2 \rangle$

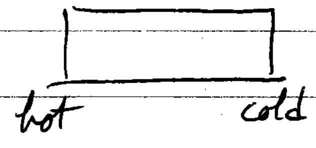
$$\langle v_x^2 \rangle = \tfrac{1}{3}\langle v^2 \rangle = \tfrac{1}{3} \frac{3 k_B T}{m}$$

(using $\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T$ equipartition theorem)

So first term is $- \frac{1}{3} m \tau \langle v^2 \rangle \frac{d\mathcal{E}}{dT} \frac{dT}{dx}$

<u>2nd term</u>: $v_x^L - v_x^R = \langle v_x \rangle$ at position x

But since thermal conductivity is usually measured in an <u>open</u> circuit



no current flows in x direction $\Rightarrow \langle v_x \rangle = 0$
So 2nd term vanishes! (see more on this later!)

$$\Rightarrow j^q = -\frac{1}{3} n \tau \langle v^2 \rangle \frac{d\mathcal{E}}{dT} \frac{dT}{dx} \quad = -\kappa \frac{dT}{dx}$$

~~now $\frac{d\mathcal{E}}{dT} \cdot n = \frac{1}{V} \frac{d\mathcal{E}}{dT}$~~

Now $\mathcal{E}$ is average energy per electron at temp T.

$\Rightarrow$ $N\mathcal{E}$ is total average energy

$\frac{N}{V}\mathcal{E} = n\mathcal{E}$ is average energy density

$\Rightarrow n \frac{d\mathcal{E}}{dT} = c_v$ specific heat per volume at constant volume

$$\kappa = \frac{1}{3} \tau c_v \langle v^2 \rangle = \frac{1}{3} \ell v c_v$$

where $\ell = \tau v$
$v = \sqrt{\langle v^2 \rangle}$

$$\sigma = \frac{ne^2\tau}{m}$$

~~$\frac{\kappa}{\sigma} = \frac{\ell v c_v}{3\left(\frac{ne^2\tau}{m}\right)}$~~

$$\frac{\kappa}{\sigma} = \frac{\frac{1}{3}\tau c_v \langle v^2 \rangle}{\frac{ne^2\tau}{m}} = \frac{m c_v \langle v^2 \rangle}{3 n e^2}$$

indep of τ!

$$\mathcal{E} = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T \qquad c_v = \frac{3}{2} n k_B$$

$$\frac{\kappa}{\sigma} = \frac{c_v k_B T}{n e^2} = \boxed{\frac{3}{2} \frac{k_B^2}{e^2} T = \frac{\kappa}{\sigma}}$$

Weidemann-Franz const is $\frac{3}{2} \frac{k_B^2}{e^2} = 1.11 \times 10^{-8}$ watt-ohm/

This is $\sim\frac{1}{2}$ the experimental value!
In his calculation Drude made a factor 2 error, so he reported a result 2.22×10^{-8} watt-ohm/$°K^2$ in excellent agreement with experiment!

This success was just luck. We will see, when we treat the ~~gas~~ electron gas quantum mechanically, that the correct $\langle v^2\rangle$ is ~ 100 times larger than Drude's classical result, but c_v is ~ 100 times smaller. So these two factors cancel to give a reasonable result, but just by accident!

Even in Drude's day it was known that something was not right since no electronic contribution to specific heat was ever found as large as $\frac{3}{2}nk_B$.

Thermo electric effect

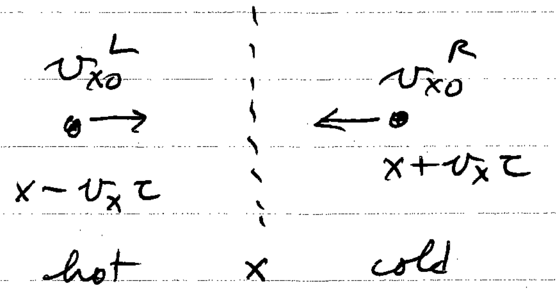
We said that $v_x^L - v_x^R = \langle v_x\rangle = 0$ in our open circuit. But since $T(x_L) > T(x_R)$ one would expect that $v_x^L > v_x^R$.

$\Rightarrow$ In steady state an electric field must be generated, in same direction as $\vec{\nabla} T$, that exactly compensates for the thermal difference in velocities, so that velocities of both right going and left going electrons are equal as they cross the same plane at x.

This is the thermoelectric field $\vec{E}$ and it is proportional to $\vec{\nabla}T$. We define the "thermopower" by

$$\vec{E} \equiv Q\vec{\nabla}T$$

To estimate Q



v_{x0}^L is from equilib thermal distrib at $T(x-v_x\tau)$
v_{x0}^R is from equilib thermal distrib at $T(x+v_x\tau)$

When the thermoelectric field $\vec{E}$ is present the speed of the right going electrons when they pass the plane at x is

$$v_x^L = v_{x0}^L - \frac{eE\tau}{m} \leftarrow \text{change in velocity due to acceleration by electric field}$$

Similarly

$$v_x^R = v_{x0}^R + \frac{eE\tau}{m}$$

So $v_x^L - v_x^R = v_{x0}^L - v_{x0}^R - \frac{2eE\tau}{m} = 0$

determines value of E

$$v_{x0}^L = v_x\left(T(x-v_x\tau)\right) = v_x(T(x)) - \frac{dv_x}{dT}\frac{dT}{dx}v_x\tau$$

$$v_{x0}^R = v_x\left(T(x+v_x\tau)\right) = v_x(T(x)) + \frac{dv_x}{dT}\frac{dT}{dx}v_x\tau$$

$$v_x^L - v_x^R = -2\frac{dv_x}{dT}\frac{dT}{dx}v_x\tau - \frac{2eE\tau}{m} = 0$$

use $\left(\frac{dv_x}{dT}\right)v_x = \frac{1}{2}\frac{dv_x^2}{dT}$

So $$E = -\frac{m}{2e}\frac{d v_x^2}{dT}\frac{dT}{dx}$$

$$\langle v_x^2\rangle = \frac{1}{3}\langle v^2\rangle \quad , \quad ~~\langle v^2\rangle = \frac{3k_BT}{m}~~$$

$$= \frac{2}{3m}\varepsilon \quad \text{where} \quad \varepsilon = \frac{1}{2}m\langle v^2\rangle$$

$$E = -\frac{1}{3e}\frac{d\varepsilon}{dT}\frac{dT}{dx}$$

$$c_v = n\frac{d\varepsilon}{dT}$$

$$E = -\frac{1}{3ne}c_v\frac{dT}{dx} = Q\frac{dT}{dx}$$

~~$c_v = \frac{3}{2}nk_B$ classically~~

$$\Rightarrow Q = -\frac{c_v}{3ne}$$

Classically $c_v = \frac{3}{2}nk_B \Rightarrow Q = -\frac{k_B}{2e}$

$$= -0.4\times 10^{-4}\ \frac{\text{volt}}{{}^\circ K}$$

observed Q is ~100 times smaller than this classical Drude result

we will get more reasonable value for Q when we use correct quantum mechanical result for c_v.