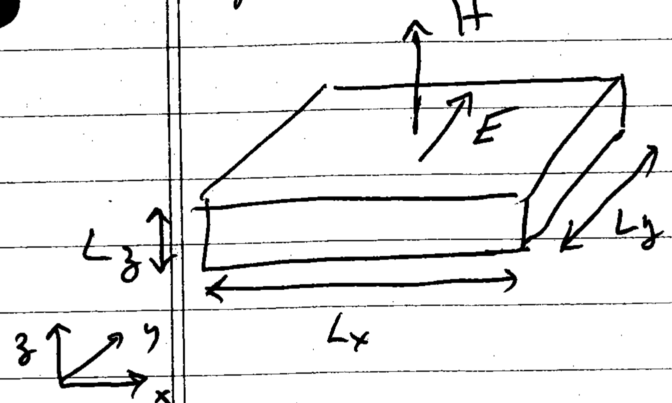
# Landau Diamagnetism - Landau Levels

First we discuss the correct quantum mechanical treatment of an electron in a static uniform magnetic field - this leads to the concept of "Landau levels" which are important for understanding: (i) Landau diamagnetism
(ii) the de Haas-von Alphen effect
(iii) the Integer Quantum Hall Effect

Consider the geometry of Hall effect and solve for exact eigenstates and eigenvalues




$\vec{H} = H\hat{z}$
$\vec{E} = E\hat{y}$
uniform fields

When computing diamagnetism and de Haas-von Alphen effect we will set $E=0$, But we will need finite E when doing the integer quantum hall effect, so we treat the general case of $E \neq 0$ now.

For a particle of charge $q$ in static electric and magnetic fields, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2 + qV$$

where $\vec{A}$ is the vector potential, $\vec{H} = \vec{\nabla}\times\vec{A}$
$V$ is the electrostatic potential, $\vec{E} = -\vec{\nabla}V$
$q = -e$ is the charge of the electron

For $\vec{H} = H\hat{z}$ we use $\vec{A} = -yH\hat{x}$
$\vec{E} = E\hat{y}$ we use $V = -Ey$

$$\Rightarrow \mathcal{H} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}yH\hat{x}\right)^2 + eEy$$

$$= \frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial z^2} - \hbar^2\frac{\partial^2}{\partial y^2} + \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{e}{c}Hy\right)^2 + 2meEy\right]$$

Try solution of form $\psi(x,y,z) = e^{ik_x x}e^{ik_z z}\phi(y)$

Substitute this form into $\mathcal{H}\psi = \varepsilon\psi$ to get

$$+\frac{1}{2m}\left[+\hbar^2k_z^2 - \hbar^2\frac{\partial^2}{\partial y^2} + \left(\hbar k_x - \frac{e}{c}Hy\right)^2 + 2meEy\right]\phi(y) = \varepsilon\phi(y)$$

or

$$\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2m}\left[\left(\hbar k_x - \frac{eH}{c}y\right)^2 + 2meEy\right]\right\}\phi(y) = \left(\varepsilon - \frac{\hbar^2k_z^2}{2m}\right)\phi(y)$$

The expression in [...] is a quadratic polynomial in the variable y. We can rewrite this term by "completing the square".

$$[\cdots] = \left(\frac{eH}{c}\right)^2 y^2 - 2\left(\frac{eH}{c}\hbar k_x - meE\right) y + \hbar^2 k_x^2$$

$$= \left(\frac{eH}{c}\right)^2 (y - y_0)^2 + \hbar^2 k_x^2 - \left(\frac{eH}{c}\right)^2 y_0^2$$

where $y_0 = \left(\frac{c}{eH}\right)\left(\hbar k_x - \frac{mcE}{H}\right)$

Now use from definition of $y_0$ above

$$\hbar k_x = \frac{eH}{c} y_0 + \frac{mcE}{H}$$

to get

$$[\cdots] = \left(\frac{eH}{c}\right)^2 (y-y_0)^2 + \overbrace{\left(\frac{eH}{c}\right)^2 y_0^2 + 2\frac{eH}{c} y_0 \frac{mcE}{H} + \frac{m^2c^2E^2}{H^2}}^{\hbar k_x}$$

$$- \left(\frac{eH}{c}\right)^2 y_0$$

$$[\cdots] = \left(\frac{eH}{c}\right)^2 (y-y_0)^2 + 2meEy_0 + \frac{m^2c^2E^2}{H^2}$$

So the Schrodinger Equation for $\phi(y)$ can now be written as

$$\left\{\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2m}\left(\frac{eH}{c}\right)^2 (y-y_0)^2 + eEy_0 + \frac{mc^2E}{2H^2}\right\}\phi(y) = \left(\varepsilon - \frac{\hbar^2 k_z^2}{2m}\right)\phi(y)$$

use cyclotron frequency $\omega_c = \frac{eH}{mc}$ to write above as:

$$\left\{ \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2} m\omega_c^2 (y-y_0)^2}_{\substack{\text{harmonic oscillator of} \\ \text{frequency } \omega_c \text{ centered} \\ \text{at position } y_0}} + \underbrace{eEy_0}_{\substack{\text{electrostatic} \\ \text{energy of} \\ \text{particle at } y_0}} + \underbrace{\frac{e^2E^2}{2m\omega_c^2}}_{\substack{\text{constant} \\ O(E/H)^2}} \right\}\phi(y) = \left(\mathcal{E} - \frac{\hbar^2 k_z^2}{2m}\right)\phi(y)$$

with $y_0 = \frac{1}{\omega_c}\left(\frac{\hbar k_x}{m} + \frac{cE}{H}\right)$

Since the energy eigenvalues of the harmonic oscillator are $\hbar\omega_c(n+1/2)$ with $n = 0, 1, 2, \ldots$, the eigenvalues of the electron in the Hall geometry are:

$$\mathcal{E}(k_x, k_z, n) = \frac{\hbar^2 k_z^2}{2m} + eEy_0 + \frac{e^2 E}{2m\omega_c} + \hbar\omega_c(n+1/2)$$

where the dependence of $\mathcal{E}$ on $k_x$ is via

$$y_0 = \frac{1}{\omega_c}\left(\frac{\hbar k_x}{m} - \frac{cE}{H}\right)$$

We first consider the special case of $E = 0$ at finite $H$.

$$\mathcal{E}(k_x, k_z, n) = \underbrace{\frac{\hbar^2 k_z^2}{2m}}_{\substack{\text{kinetic energy} \\ \text{of motion parallel} \\ \text{to magnetic field } H\hat{z}}} + \underbrace{\hbar\omega_c(n+1/2)}_{\substack{\text{energy of motion in} \\ xy \text{ plane transverse} \\ \text{to magnetic field } H\hat{z}}}$$

Let us focus on the two dimensional (2D) problem of motion in the xy plane

Define $\tilde{\varepsilon} \equiv \varepsilon - \frac{\hbar^2 k_z^2}{2m} = \hbar\omega_c(n+1/2) \qquad n = 0, 1, 2, \ldots$

If $\vec{H} = 0$, we know that the eigenstates in the xy plane are just plane waves with wavevector $\vec{k} = k_x\hat{x} + k_y\hat{y}$, and a continuous (as $L \to \infty$) eigenvalue spectrum

$$\tilde{\varepsilon} = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m}$$

From problem (3b) of HW set 1, you found that the density of states for this 2D free electron gas ($\vec{H}=0$) was a <u>constant</u>

$$g_{2D}(\tilde{\varepsilon}) = \frac{m}{\pi\hbar^2}$$

Now we see that when a finite $\vec{H}$ is applied, the continuous energy spectrum breaks up into a set of degenerate discrete levels labeled by the integer index $n$, $\tilde{\varepsilon} = \hbar\omega_c(n+1/2)$.

For fixed $n$, there are many degenerate eigenstates labeled by the different values of $k_x$ (when $\vec{E} = 0$, $\tilde{\varepsilon}$ is independent of $k_x$). The set of degenerate states labeled by a common value of $n$ is called the $n$th "Landau level". The energy spacing between Landau levels is $\hbar\omega_c \propto H$ increases and gets large as $H$ increases.

Eigenstates are the wavefunctions

$$\psi_{k_x, k_z, n}(x, y, z) = e^{i k_z z} e^{i k_x x} \phi_n(y - y_0)$$

with $y_0 = \frac{\hbar k_x}{m\omega_c}$ (when $\vec{E} = 0$)

$n^{th}$ harmonic oscillator eigenfunction, centered at $y_0$

Let us now find the degeneracy of each Landau level. This degeneracy will be given by the number of ways one can choose the value of $k_x$.

There are two conditions that restrict the allowed values of $k_x$. Taking periodic boundary conditions as we have done before requires that

$$k_x = \frac{2\pi}{L_x} \times (\text{integer})$$

So the spacing between allowed values of $k_x$ is $\Delta k = \frac{2\pi}{L_x}$

But also, since our wavefunction $\psi$ is centered in the $y$ direction about $y_0$, we must have that $y_0$ is contained within the boundaries of the system, ie

$$0 \le y_0 \le L_y$$

This condition puts a maximum allowed value on $k_x$

$$0 \le y_0 \le L_y \Rightarrow 0 \le \frac{\hbar k_x}{m\omega_c} \le L_y \Rightarrow k_{x\,max} = \frac{L_y m\omega_c}{\hbar}$$

The number of allowed values of $k_x$ is then

$$N = \frac{k_{x\,max}}{\Delta k} = \frac{\left(\frac{m\omega_c L_y}{\hbar}\right)}{\left(\frac{2\pi}{L_x}\right)} = \frac{m\omega_c}{2\pi\hbar} L_x L_y = \frac{m}{2\pi\hbar} \frac{eH}{mc} L_x L_y$$

If we include an extra factor 2 for spin degeneracy (ignoring now that in principle the spin $\vec{s}$ will couple to $\vec{H}$) the degeneracy of each Landau level is

$$\frac{2m}{2\pi\hbar}\frac{eH}{mc}L_xL_y = \frac{2e}{hc}\Phi \qquad (h = 2\pi\hbar)$$

where $\Phi = HL_xL_y$ is the total magnetic flux through the system. Since the degeneracy above is a pure number, we see that $\frac{hc}{2e}$ has units of magnetic flux and one defines

$$\Phi_0 \equiv \frac{hc}{2e} = 2.07\times 10^{-7}\ \text{gauss-cm}^2$$

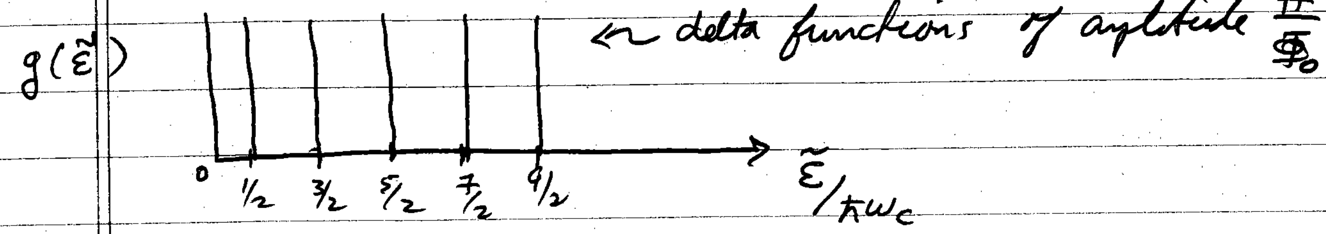
called the <u>flux quantum</u>

$\Rightarrow$ degeneracy of each Landau level is

$$(\Phi/\Phi_0)$$

We can summarize the above results in terms of a <u>density of states</u> for the energy $\tilde{\varepsilon}$. The number of ~~values~~ states per unit area per energy $\tilde{\varepsilon}$ is $g(\tilde{\varepsilon})$, which will be a set of $\delta$-functions at the energies $\tilde{\varepsilon} = \hbar\omega_c(n+1/2)$. The amplitude of each $\delta$-function is the degeneracy of each Landau level <u>per unit area</u> ie $\frac{\Phi}{\Phi_0}\frac{1}{L_xL_y} = \frac{H}{\Phi_0}$

$$g(\tilde{\varepsilon}) = \sum_n \delta\left(\tilde{\varepsilon} - \hbar\omega_c(n+1/2)\right)\frac{H}{\Phi_0}$$




We can compute the <u>average density</u> of states by averaging the above over an energy interval $\Delta\varepsilon$.

average density of states $\bar{g} = \dfrac{(\#\ \delta\text{-function spikes in } \Delta\varepsilon) \times \frac{H}{\Phi_0}}{\text{interval width } \Delta\varepsilon}$

take $\Delta\varepsilon = \hbar\omega_c$ the width between spikes

$\Rightarrow$ one spike per interval $\Delta\varepsilon$

$$\bar{g}(\tilde{\varepsilon}) = \frac{\left(\frac{H}{\Phi_0}\right)}{\hbar\omega_c} = \frac{H}{\left(\frac{hc}{2e}\right)} \frac{1}{\hbar\left(\frac{eH}{mc}\right)} = \frac{m}{\pi\hbar^2}$$

using $h = 2\pi\hbar$

From problem (3b) of HW set 1, you will find that the density of states in 2D for a free electron gas with $E = 0$, $H = 0$, is just the constant

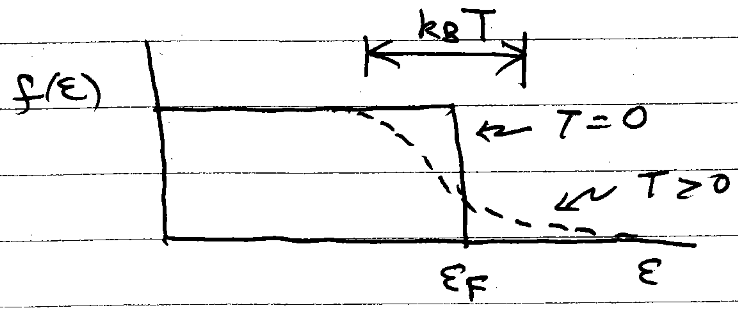
$$g(\tilde{\varepsilon}) = \frac{m}{\pi\hbar^2}$$

So we see that turning on the magnetic field bunches the states up into degenerate discrete levels, but the average number of states per unit energy remains the same

Suppose we had an actual 2D electron gas.
One can think of making this in a thin metallic film or a semiconductor inversion layer where the gas is confined to a region in space along $\hat{z}$ so small that only the lowest allowed value of $k_z$ is occupied, ie $\frac{2\pi}{L_z} = \Delta k_z$ gives $\frac{\hbar^2(\Delta k_z)^2}{2m}$ larger than all other energy scales.

What is necessary so that one could detect the difference between the discrete Landau level structure at finite $H > 0$, and the average density of states which is equal to its $H=0$ value?

If $f$ is the Fermi function, we know that finite temperature smears out the sharp cutoff at $\varepsilon = \varepsilon_F$ that exists at $T=0$.




To see the Landau level structure we thus need this smearing to be small on the scale of the spacing between the Landau levels

ie. need $k_B T \ll \hbar\omega_c$

using $\omega_c = \frac{eH}{mc}$ and $m$ the free electron mass one can compute

$$\omega_c = 1.76 \times 10^{11}\ \text{sec}^{-1} \quad \text{for a } H = 1 \text{ tesla} = 10^4 \text{ gauss}$$

magnetic field.

1 tesla is a big field. In a laboratory setup such as in BL one can buy a 10 tesla magnet. Larger field strengths require specialized facilities

So for $H = 1$ tesla, $\boxed{\frac{\hbar\omega_c}{k_B} = 1.34\ {}^\circ K}$

So in a 1 tesla field one needs to go well below $1^\circ K$ to see Landau level structure. In a 10 tesla field one needs to go well below $10^\circ K$. So quite low temperatures are needed.

There is a second condition. In solving Schrodinger's equation for the Landau levels, we ignored any sources of electron scattering (scattering off phonons, plasmons, lattice impurities, etc.) If $\tau$ is the scattering time, including such scattering generally leads, via the uncertainty principle, to a broading of the energy levels of the eigenstates to a finite width $\delta E \sim \frac{\hbar}{\tau}$

So to see Landau level structure we need

$$\delta\varepsilon \ll \hbar\omega_c \Rightarrow \frac{\hbar}{\tau} \ll \hbar\omega_c$$

$$\Rightarrow \omega_c \tau \gg 1$$

using $\omega_c = 1.76 \times 10^{11}\ \mathrm{sec}^{-1}$ in $H = 1$ tesla
and from resistivity measurements used to estimate
$\tau$ from Drude's model we get

room temp $\tau \sim 10^{-14}$ sec, $\omega_c \tau \sim 0.00176$
$77^\circ K$ (liquid N) $\tau \sim 10^{-13}$ sec, $\omega_c \tau \sim 0.0176$

we again see that we will need very low temperatures (large $\tau$) to get $\omega_c \tau \gg 1$.

Landau level structure is typically only observable if one goes down to liquid Helium temperatures $\sim 5^\circ K$.