Energy. Let us define

$$\frac{E(\varepsilon)}{V} = \int_0^{\varepsilon} d\varepsilon' \, g(\varepsilon')\,\varepsilon'$$

Then the total ground state energy will be $E(\varepsilon_F)$

$\underline{H=0}$ $$g^0(\varepsilon) = \frac{3}{2}\frac{n}{\varepsilon_F^0}\sqrt{\frac{\varepsilon}{\varepsilon_F^0}}$$

$$\frac{E^0(\varepsilon)}{V} = \frac{3}{2}\frac{n}{(\varepsilon_F^0)^{3/2}}\int_0^{\varepsilon} d\varepsilon' \, \varepsilon'^{3/2}$$

$$= \frac{3}{2}\cdot\frac{2}{5}\frac{n}{(\varepsilon_F^0)^{3/2}}\,\varepsilon^{5/2}$$

$$\boxed{\frac{E^0(\varepsilon)}{V} = \frac{3}{5}\, n \left(\frac{\varepsilon}{\varepsilon_F^0}\right)^{3/2}\varepsilon}$$

from above we get familiar result

$$\frac{E^0(\varepsilon_F^0)}{V} = \frac{3}{5}\, n\, \varepsilon_F^0 \quad \text{at } H=0$$

$\underline{H>0}$

$$g(\varepsilon) = \frac{3}{2}\frac{n}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\;\frac{1}{2}\sum_n{}' \frac{1}{\sqrt{x-n-1/2}}$$

term n appears in sum only if $x \geq n+1/2$

$$\frac{E(\varepsilon)}{V} = \int_0^{\varepsilon'} d\varepsilon' \, g(\varepsilon')\,\varepsilon' = (\hbar\omega_c)^2\int_0^x dx' \, g(x')\,x'$$

$$= \frac{3}{2}\frac{n}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\,(\hbar\omega_c)^2\sum_n{}' \frac{1}{2}\int_0^x dx' \frac{x'}{\sqrt{x'-n-1/2}}$$

as we explained in discussion concerning the computation of $G(\varepsilon)$, we can rewrite the sum and integrals as

$$\frac{E(\varepsilon)}{V} = \frac{3}{2}\frac{n}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\,(\hbar\omega_c)^2 \sum_{n=0}^{n_{max}} \frac{1}{2}\int_{n+1/2}^{x} dx' \, \frac{x'}{\sqrt{x'-n-1/2}}$$

where n_{max} is largest integer such that $x \geq n_{max}+\frac{1}{2}$

We can look up the integral in a table to find

$$\int dx \, \frac{x}{\sqrt{x-a}} = \frac{2}{3}(x+2a)\sqrt{x-a}$$

So

$$\frac{E(\varepsilon)}{V} = \frac{n}{2\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\,(\hbar\omega_c)^2 \sum_{n=0}^{n_{max}} \Big[(x'+2n+1)\sqrt{x'-n-1/2}\Big]_{n+1/2}^{x}$$

$$\frac{E(\varepsilon)}{V} = \frac{n}{2\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\,(\hbar\omega_c)^2 \sum_{n=0}^{n_{max}} (x+2n+1)\sqrt{x-n-1/2}$$

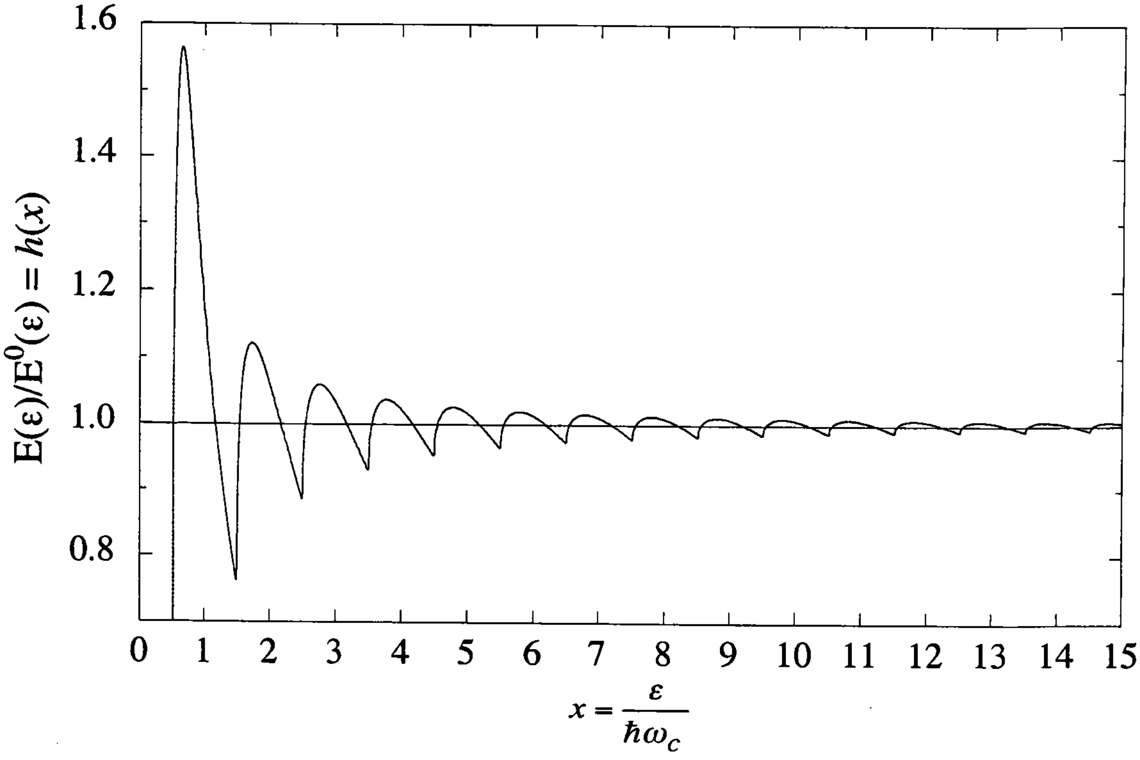
Compare to $\frac{E^0(\varepsilon)}{V} = \frac{3}{5}\, n\left(\frac{\varepsilon}{\varepsilon_F^0}\right)^{3/2}\varepsilon$ and we have

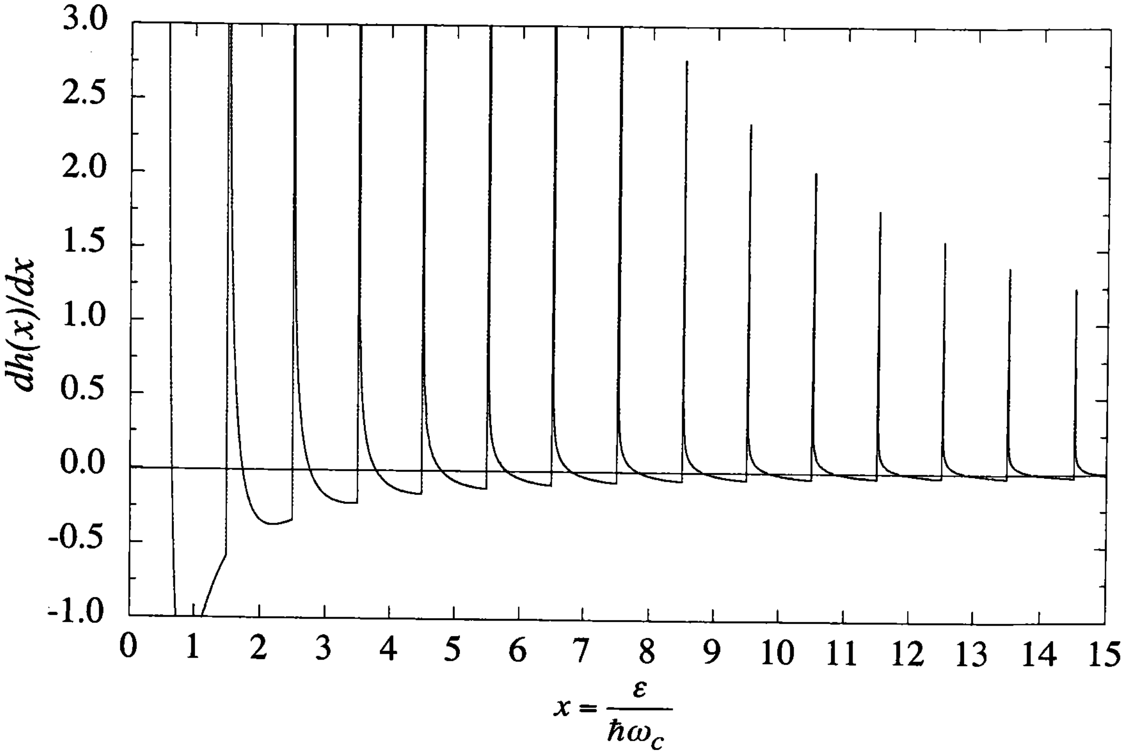
$$\boxed{\frac{E(\varepsilon)}{E^0(\varepsilon)} = \frac{5}{6}\frac{1}{x^{5/2}} \sum_{n=0}^{n_{max}} (x+2n+1)\sqrt{x-n-1/2}}$$

$x = \varepsilon/\hbar\omega_c$ and n_{max} such that $x \geq n_{max}+1/2$

define $h(x) \equiv \frac{5}{6}\frac{1}{x^{5/2}} \sum_{n=0}^{n_{max}} (x+2n+1)\sqrt{x-n-1/2}$

then $E(\varepsilon) = E^0(\varepsilon)\, h(x)$


1.6
1.4
1.2
1.0
0.8
$E(\varepsilon)/E^0(\varepsilon) = h(x)$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x = \frac{\varepsilon}{\hbar\omega_c}$


3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
$dh(x)/dx$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x = \frac{\varepsilon}{\hbar\omega_c}$

The ground state energy for finite $H>0$ is now $E(\varepsilon_F)$ where $\varepsilon_F = \varepsilon_F^0 + \delta\varepsilon$

$$E(\varepsilon_F) = E^0(\varepsilon_F^0 + \delta\varepsilon)\, h(x^0 + \delta x) \qquad \begin{cases} x^0 = \dfrac{\varepsilon_F^0}{\hbar\omega_c} \\ \delta x = \dfrac{\delta\varepsilon}{\hbar\omega_c} \end{cases}$$

Since $\delta\varepsilon$ is small, expand to lowest order

$$E(\varepsilon_F) \simeq \left[E^0(\varepsilon_F^0) + \frac{dE^0(\varepsilon_F^0)}{d\varepsilon}\delta\varepsilon\right]\left[h(x^0) + h'(x^0)\frac{\delta\varepsilon}{\hbar\omega_c}\right]$$

$$\simeq E^0(\varepsilon_F^0)h(x^0) + \left[E^0(\varepsilon_F^0)\frac{h'(x^0)}{\hbar\omega_c} + \frac{dE^0(\varepsilon_F^0)}{d\varepsilon}h(x^0)\right]\delta\varepsilon$$

using $\dfrac{E^0(\varepsilon)}{V} = \dfrac{3}{5}\, n\, \dfrac{\varepsilon^{5/2}}{(\varepsilon_F^0)^{3/2}}$ we get

$$\frac{1}{V}\frac{dE^0}{d\varepsilon} = \frac{3}{2}\, n\, \frac{\varepsilon^{3/2}}{(\varepsilon_F^0)^{3/2}} \quad \text{so} \quad \frac{dE^0(\varepsilon_F^0)}{d\varepsilon} = \frac{3}{2}\, n V$$

also $\dfrac{E^0}{V} = \dfrac{3}{5}\, n\, \varepsilon_F^0$ so $\dfrac{dE^0(\varepsilon_F^0)}{d\varepsilon} = \dfrac{5}{2}\dfrac{E^0}{\varepsilon_F^0}$

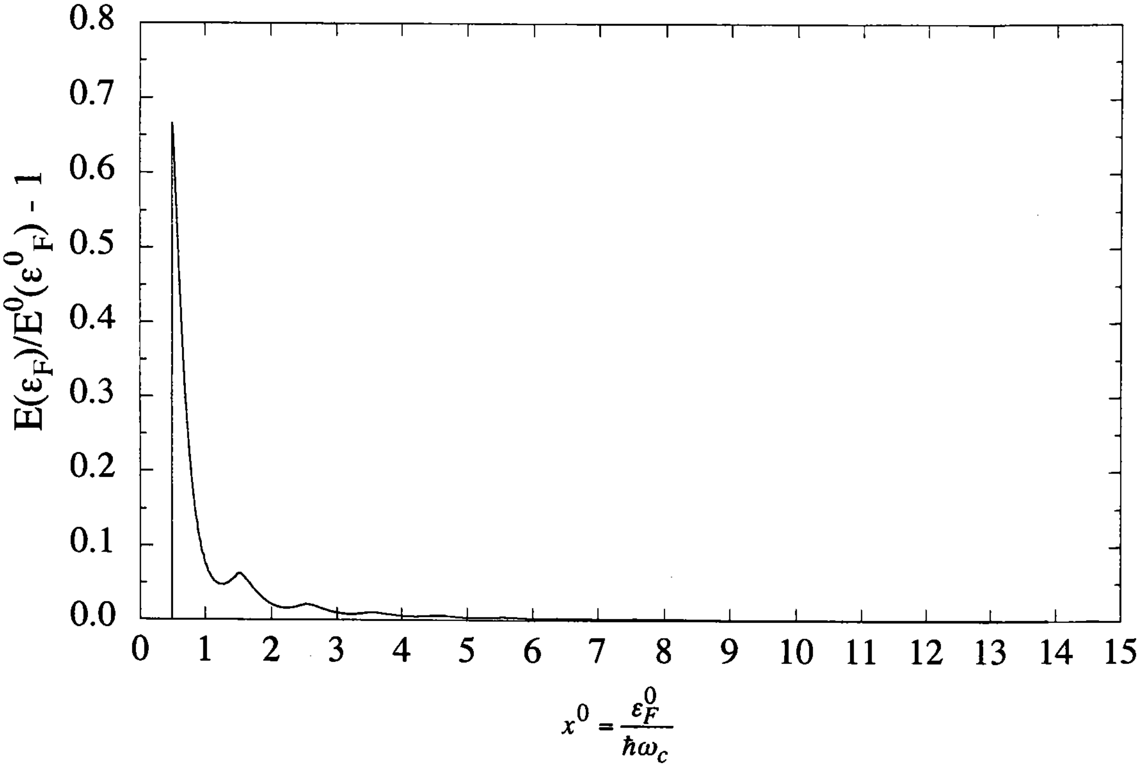
$$E(\varepsilon_F) = E^0(\varepsilon_F^0)\left[h(x^0) + \left(\frac{h'(x^0)}{\hbar\omega_c} + \frac{5}{2}\frac{h(x^0)}{\varepsilon_F^0}\right)\delta\varepsilon\right]$$

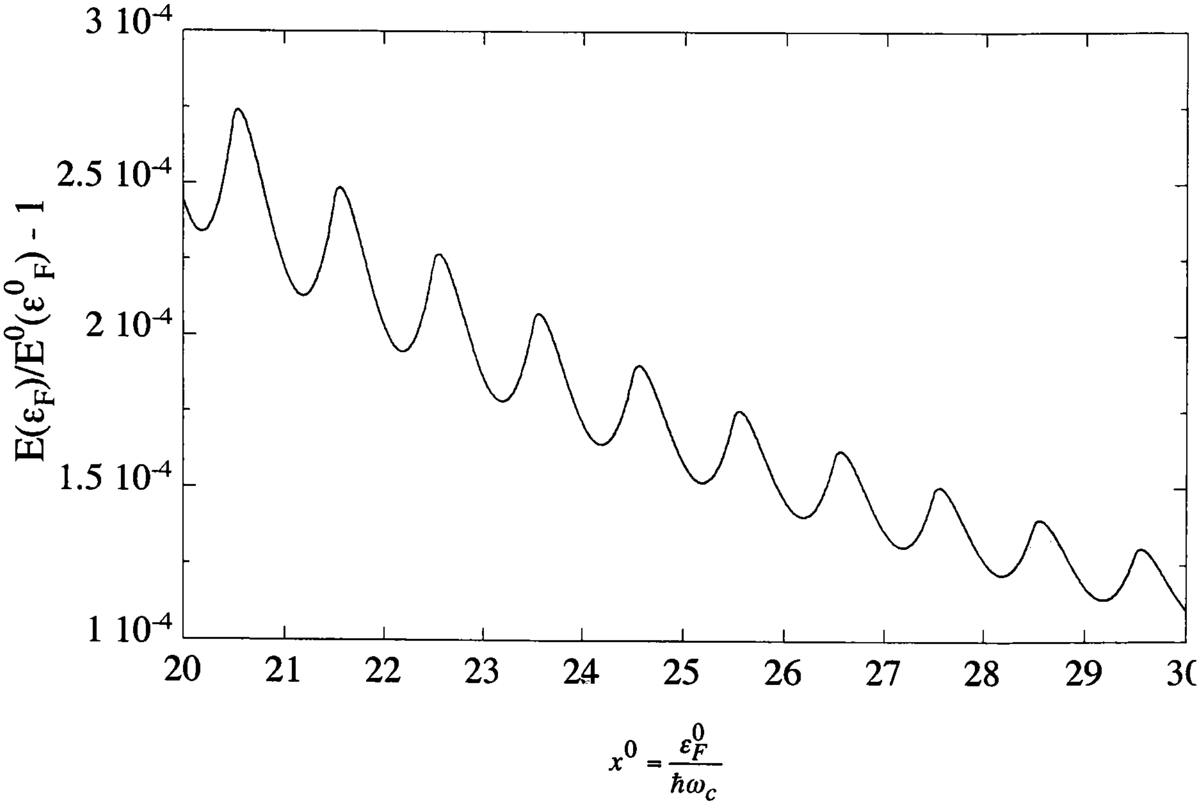
$$\boxed{E(\varepsilon_F) = E^0(\varepsilon_F^0)\left[h(x^0) + \left(h'(x^0) + \frac{5}{2}\frac{h(x^0)}{x^0}\right)\delta x\right] \quad \text{where } x^0 = \varepsilon_F^0/\hbar\omega_c}$$

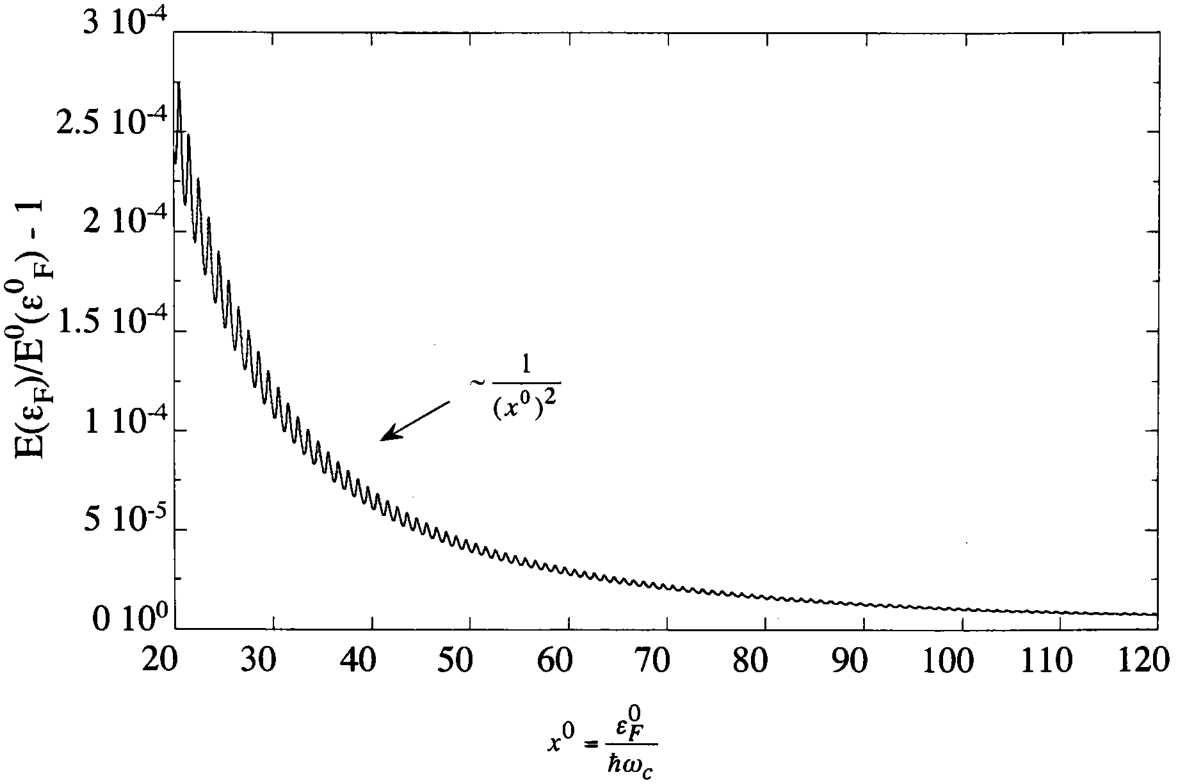
with $\delta x = \dfrac{1 - f(x^0)}{f'(x^0) + \frac{3}{2}\frac{f(x^0)}{x^0}}$ as found earlier

We can now plot $\dfrac{E(\varepsilon_F)}{E^0(\varepsilon_F^0)}$ as function of $x^0 = \dfrac{\varepsilon_F^0}{\hbar\omega_c}$

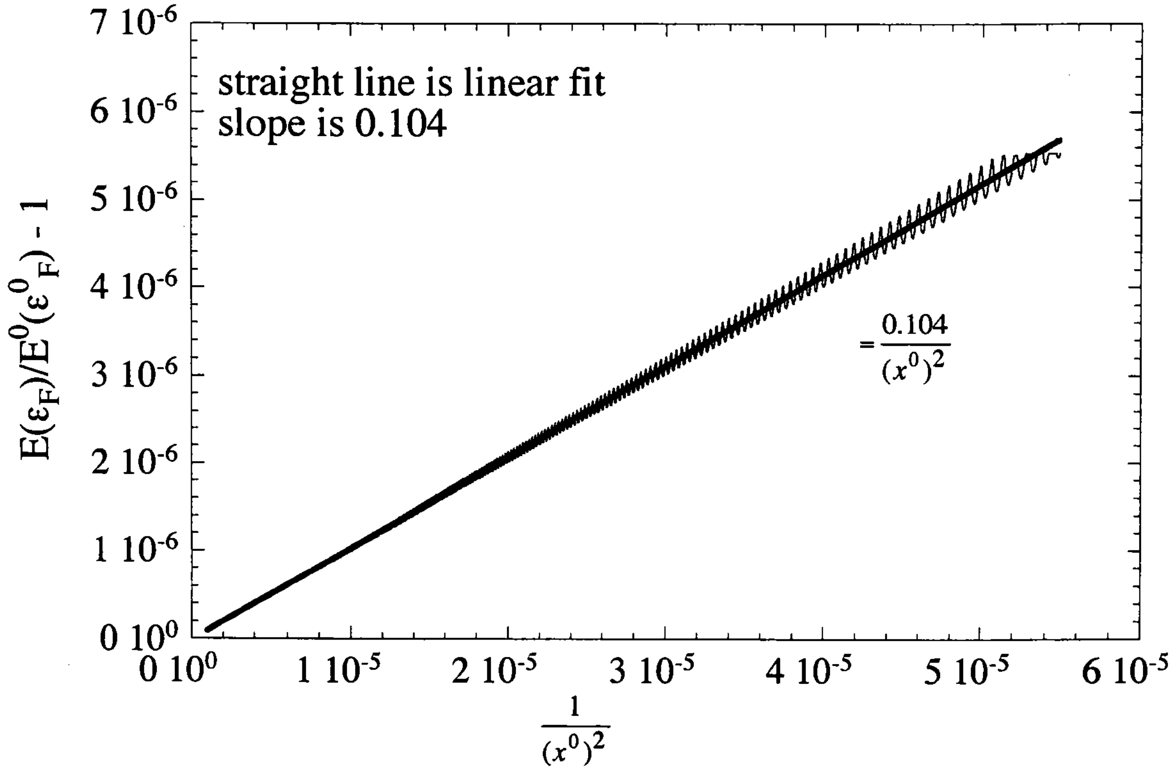
Since we expect $E(\varepsilon_F)/E^0(\varepsilon_F^0) \to 1$ as $x^0 \to \infty$, we plot $\left[\dfrac{E(\varepsilon_F)}{E(\varepsilon_F^0)} - 1\right]$


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$E(\varepsilon_F)/E^0(\varepsilon^0_F) - 1$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$

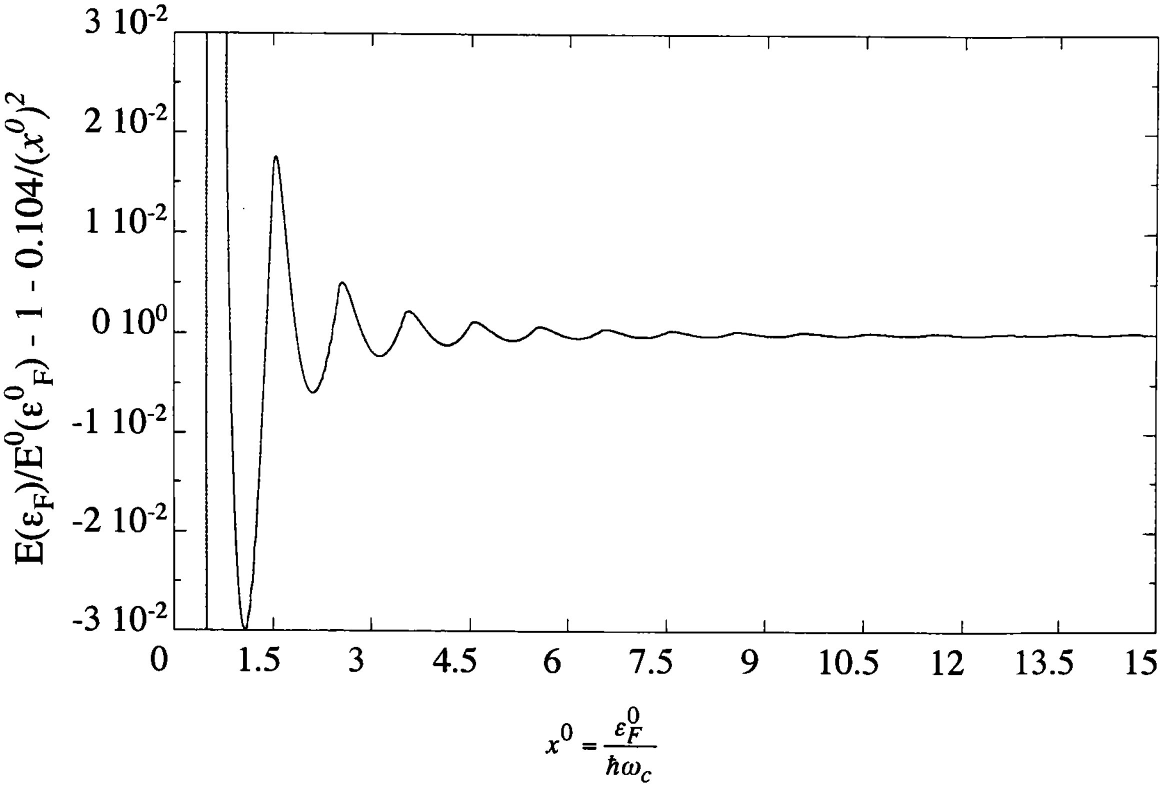

3 10-4
2.5 10-4
2 10-4
1.5 10-4
1 10-4
$E(\varepsilon_F)/E^0(\varepsilon^0_F) - 1$
20 21 22 23 24 25 26 27 28 29 30
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$



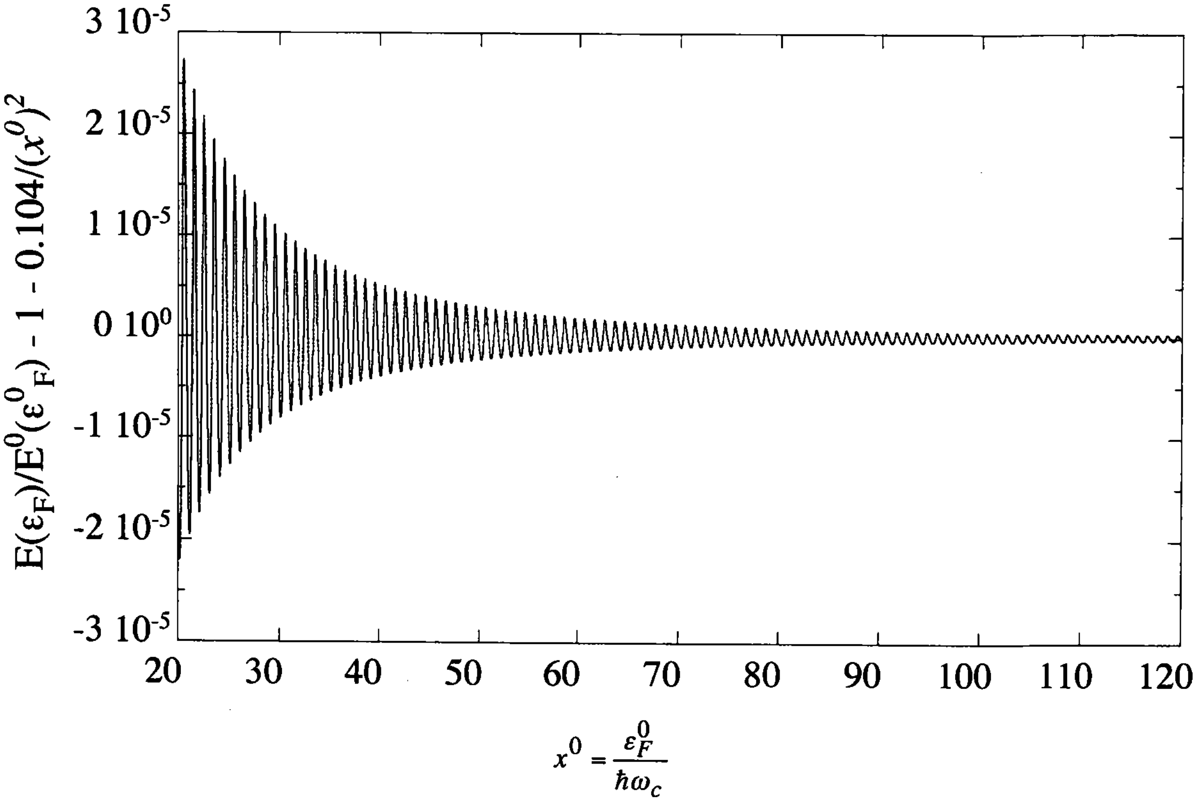
3 10-4
2.5 10-4
2 10-4
1.5 10-4
1 10-4
5 10-5
0 10^0
20 30 40 50 60 70 80 90 100 110 120
E(ε_F)/E^0(ε^0_F) - 1
$\sim \frac{1}{(x^0)^2}$
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$



7 10-6
6 10-6
5 10-6
4 10-6
3 10-6
2 10-6
1 10-6
0 10^0
0 10^0 1 10-5 2 10-5 3 10-5 4 10-5 5 10-5 6 10-5
E(ε_F)/E^0(ε^0_F) - 1
straight line is linear fit
slope is 0.104
$= \frac{0.104}{(x^0)^2}$
$\frac{1}{(x^0)^2}$



3 10-2
2 10-2
1 10-2
0 10^0
-1 10-2
-2 10-2
-3 10-2
$E(\varepsilon_F)/E^0(\varepsilon^0_F) - 1 - 0.104/(x^0)^2$
0
1.5
3
4.5
6
7.5
9
10.5
12
13.5
15
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$



3 10-5
2 10-5
1 10-5
0 10^0
-1 10-5
-2 10-5
-3 10-5
$E(\varepsilon_F)/E^0(\varepsilon^0_F) - 1 - 0.104/(x^0)^2$
20
30
40
50
60
70
80
90
100
110
120
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$

From our plot of $\frac{E(\varepsilon_F)}{E^0(\varepsilon_F^0)} - 1$ we conclude that

$E(\varepsilon_F)$ has the form

$$E(\varepsilon_F) = E^0(\varepsilon_F^0)\left[1 + \frac{\alpha}{(x^0)^2} + g(x^0)\right]$$

where $\alpha = 0.104$ and $g(x)$ is a small oscillating part with period $\Delta x = 1$ and amplitude that vanishes as $x^0 \to \infty$.

At high temperature such that $k_B T \gg \hbar\omega_c$ (in practice this is anything more than a few °K) thermal fluctuations smear out the oscillations $g(x)$ and average them to zero. One is left with

$$E(\varepsilon_F) = E^0(\varepsilon_F^0) + \frac{\alpha E^0(\varepsilon_F^0)}{(x^0)^2}$$

since $x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$ and $\omega_c = \frac{eH}{mc}$

the second term is $\propto H^2$ and so this is the term that gives the Landau diamagnetic susceptibility

$$E(\varepsilon_F) = E^0(\varepsilon_F^0) + \alpha\left(\frac{3}{5} n \varepsilon_F^0 V\right)\left(\frac{\hbar e H}{mc\,\varepsilon_F^0}\right)^2$$

where we used $E^0(\varepsilon_F^0) = \frac{3}{5} n \varepsilon_F^0 V$ V is volume

$$E(\varepsilon_F) = E^0(\varepsilon_F^0) + \frac{3}{5}\alpha \frac{m}{\varepsilon_F^0} V \frac{\hbar^2 e^2}{m^2 c^2} H^2$$

recall the Bohr magneton $\mu_0 = \frac{e\hbar}{2mc}$

So $E(\varepsilon_F) = E^0(\varepsilon_F^0) + \frac{12}{5}\alpha \frac{m V}{\varepsilon_F^0} \mu_0^2 H^2$

So $\chi_L = -\frac{1}{V}\frac{\partial^2 E}{\partial H^2}\Big|_{H=0} = -\frac{24}{5}\alpha \frac{m}{\varepsilon_F^0}\mu_0^2$

Recall that the Pauli paramagnetic susceptibility (due to intrinsic electron magnetic moment) was

$$\chi_p = \frac{3}{2}\frac{m}{\varepsilon_F^0}\mu_0^2$$

So

$\chi_L = \frac{-16}{5}\alpha\, \chi_p$ using $\alpha = .104$ we get

$\chi_L = -0.333\, \chi_p$ since $\chi_p > 0$ paramagnetic $\Rightarrow \chi_L < 0$ diamagnetic

Landau's exact result gives $\alpha = \frac{5}{48}$

Now Landau, in his original calculation, did not have such nice computers! So he found a different method, involving a finite temperature calculation of the free energy, and using various integral approximates to discrete sums. This way he arrived at the analytical result

$$\boxed{\chi_L = -\frac{1}{3}\chi_p}$$

certainly in agreement with our numerical result.

In the preceeding calculations, we treated the paramagnetic and diamagnetic effects separately. ie, when computing Pauli paramagnetism we ignore the change in electron wavefunction due to the presence of the magnetic field H, and only considered the interaction of H with the intrinsic electron magnetic moment $\mu_0 s$. When computing Landau diamagnetism we ignored this interaction with the intrinsic moment, and considered only the effect of H on the eigenstates and hence the density of states.

Of course both effects are there simultaneously, so the total magnetic susceptibility of the free electron gas is therefore

$$\chi = \chi_p + \chi_L = \chi_p - \frac{1}{3}\chi_L = \frac{2}{3}\chi_p$$

Since $\chi_p > 0$, the net effect is paramagnetic

For some more traditional calculations of Landau diamagnetism see:

Notes from AP Young UC Santa Cruz
http://bartok.ucsc.edu/peter/231/magnetic_field/node5.htm

Pathria - "Statistical Mechanics", pgs 206 - 209

Landau + Lifshitz - "Statistical Mechanics V1", pgs 172 - 175

The de Haas - van Alphen effect

At sufficiently low temperature and high magnetic field, so that $\hbar\omega_c > k_B T$, the oscillations due to the discrete Landau levels can be observed in measurements of magnetization $M = -\frac{1}{V}\partial E/\partial H$. These were first observed by de Haas and van Alphen in 1930 in magnetization measurements on Bi at 14.2°K. Similar oscillations are found in susceptibility $\chi = \partial M/\partial H$, conductivity (Shubnikov - de Haas effect), and many other quantities. Since we found that ε_F has such oscillations, so $g(\varepsilon_F)$ will have such oscillations, hence we can easily see why many physical quantities also oscillate.

The period of oscillations is in the <u>inverse</u> magnetic field $1/H$

period is $\Delta X^0 = 1 \Rightarrow \Delta\left(\frac{\varepsilon_F^0}{\hbar\omega_c}\right) = 1 \qquad \omega_c = \frac{eH}{mc}$

since ε_F^0 is fixed while H varies, we have oscillations that are periodic in $1/H$ with period

$$\Delta\left(\frac{1}{H}\right) = \frac{\hbar}{\varepsilon_F^0}\frac{e}{mc}$$

we can rewrite this as

$$\Delta\left(\frac{1}{H}\right) = \frac{\hbar\, 2m}{\hbar^2 k_F^2}\,\frac{e}{mc} = \frac{2e}{\hbar c\, k_F^2}$$

cross sectional area of the Fermi sphere is $A_F = \pi k_F^2$, so

$$\boxed{\Delta\left(\frac{1}{H}\right) = \frac{2\pi e}{\hbar c}\,\frac{1}{A_F}}$$

The above turns out to be more generally true. For electrons in a periodic potential (as opposed to our free electron model) the Fermi surface is not necessarily a sphere. Still the above relation holds where A_F is the extremal ~~maximal~~ cross sectional area of the Fermi surface perpendicular to the direction of the applied magnetic field. The de Haas - van Alphen effect thus became one of the methods for measuring the shape of the Fermi surface.

see Ashcroft + Mermin Chpt 14 for more details