Integer Quantum Hall Effect (IQHE)

1985 Nobel prize to Klaus von Klitzing for expt discovery
(1998 Nobel prize to Laughlin, Störmer and Tsui for
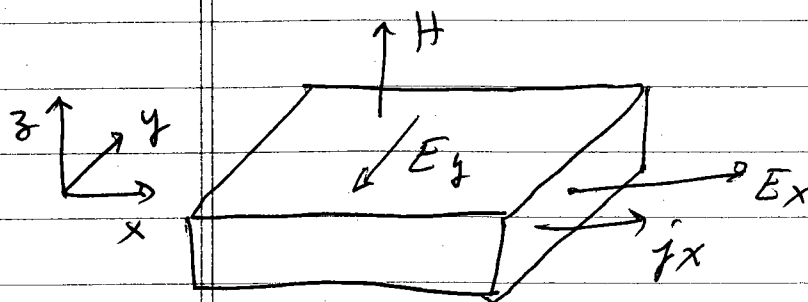fractional quantum Hall effect)

Refs: Kittel 8th ed pg 499

K. Huang — Statistical Physics, 2nd ed, pg 261

K. von Klitzing, Rev. Mod. Phys. 58, 519 (1986)

R.B. Laughlin, PRB 23, 5632 (1981)

S.M. Girvin, www.arxiv.org preprint condmat/9907002

Consider the geometry of the Hall effect

current confined to
flow along $\hat{x}$ but
$\vec{H}$ results in component
of $\vec{E}$ along $\hat{y}$

In general, the current $\vec{j}$ is <u>not</u> parallel to the
total electric field $\vec{E}$. We can formulate this
in terms of a <u>conductivity tensor</u> $\overleftrightarrow{\sigma}$

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E}$$
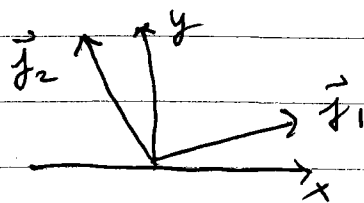
considering just the xy plane:

$$\begin{pmatrix} \dot{j}_x \\ \dot{j}_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

For a system with rotational invariance, $\overleftrightarrow{\sigma}$ must satisfy some constraints on its components.

For $\vec{E}_1 = \begin{pmatrix} E \\ 0 \end{pmatrix}$ we have $\vec{j}_1 = \overleftrightarrow{\sigma} \cdot \vec{E}_1 = \begin{pmatrix} \sigma_{xx} E \\ \sigma_{yx} E \end{pmatrix}$

For $\vec{E}_2 = \begin{pmatrix} 0 \\ E \end{pmatrix}$ we have $\vec{j}_2 = \overleftrightarrow{\sigma} \cdot \vec{E}_2 = \begin{pmatrix} \sigma_{xy} E \\ \sigma_{yy} E \end{pmatrix}$

But $\vec{E}_2$ is just a $90°$ rotation of $\vec{E}_1$, $\begin{array}{c} E_2 \uparrow \\ \searrow E_1 \end{array}$

So we expect that $\vec{j}_2$ must be a $90°$ rotation of $\vec{j}_1$

$\Rightarrow \begin{pmatrix} j_{2x} \\ j_{2y} \end{pmatrix} = \begin{pmatrix} -j_{1y} \\ j_{1x} \end{pmatrix}$

$\Rightarrow \sigma_{xy} = -\sigma_{yx}$ and $\sigma_{yy} = \sigma_{xx}$

So conductivity tensor must be of the form

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

The <u>resistivity tensor</u> is the inverse of the conductivity tensor

$$\overleftrightarrow{\rho} = \overleftrightarrow{\sigma}^{-1} \qquad \text{so} \qquad \vec{E} = \overleftrightarrow{\rho} \cdot \vec{j}$$

$$\overleftrightarrow{\rho} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} \qquad \text{with} \qquad \rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2}$$

$$\rho_{xy} = \frac{-\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}$$

check that this ρ_{xx} and ρ_{xy} give $\overleftrightarrow{\rho} \cdot \overleftrightarrow{\sigma} = \overleftrightarrow{I}$ identity tensor

From the above we can see that some of our
intuition, derived from the $H=0$ case, does
not hold when $H>0$.

When $\vec{H}=0$, $\overleftrightarrow{\sigma}$ is diagonal, $\sigma_{xy}=0$

$\Rightarrow \overleftrightarrow{\rho}$ is diagonal and $\rho_{xx} = \frac{1}{\sigma_{xx}}$
resistivity is inverse of conductivity.

If we have a system where $\sigma_{xx}=0$, then
$\rho_{xx} \to \infty$. System is insulating with infinite
resistivity. $\sigma_{xx}=0 \Rightarrow$ system is $\underline{dissipative}$
(Joule heating I^2R is huge)

When $\vec{H}>0$ $\qquad \sigma_{xy} \neq 0 \qquad \rho_{xx} \neq \frac{1}{\sigma_{xx}}$

If we have a system with $\sigma_{xx}=0$ then
$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2+\sigma_{xy}^2} = 0!$ (not infinite as when $\vec{H}=0$)

$$\vec{E} = \begin{pmatrix} 0 & \rho_{xy} \\ -\rho_{xy} & 0 \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \rho_{xy}\,j_y \\ -\rho_{xy}\,j_x \end{pmatrix}$$

and we have $\vec{j} \cdot \vec{E} = 0!$ Current is $\perp$ electric field
Consider an electron carrying this current.
The force on the electron is $\vec{F} \propto \vec{E}$, the
velocity of the electron is $\vec{v} \propto \vec{j}$. So
$\vec{j} \cdot \vec{E} = 0 \Rightarrow \vec{F} \cdot \vec{v} = 0$ so no work done
on electron. Current flows $\underline{without\ dissipation}$.

So when $H = 0$, then $\sigma_{xx} = 0$ is a <u>dissipative</u> state but when $H > 0$, then $\sigma_{xx} = 0$ is a <u>dissipationless</u> state!

In the quantum Hall effect, we will find that at certain discrete values of H, we have

$$\begin{cases} \sigma_{xx} = \rho_{xx} = 0 \\ \sigma_{xy} = -\dfrac{1}{\rho_{xy}} = \dfrac{\nu e^2}{h} \end{cases}$$

where in the <u>integer</u> quantum Hall effect, ν is an integer while in the <u>fractional</u> quantum Hall effect, ν is a simple rational fraction.

The quantum Hall state thus describes a dissipationless current flow in which the Hall conductance σ_{xy} is quantized in units of e^2/h.

The <u>fractional</u> quantum Hall effect depends on understanding the effects of the interactions between electrons — we will not discuss it.

The <u>integer</u> quantum Hall effect can be understood in terms of non-interacting electrons. This is the case we will consider.

Quantization of σ_{xy} happens independent of details of system — it happens for real, dirty, experimental systems! So it reflects a deep fact about quantum nature of the system

Conductivity and resistivity tensors within Drude model

We can now recast our discussion of the Hall effect, as computed within the Drude model, in terms of the conductivity and resistivity tensors.

Drude's equ of motion for momentum $\vec{p} = m\vec{v}$

$$\frac{d\vec{p}}{dt} = \vec{F} - \frac{\vec{p}}{\tau} \qquad \text{in steady state} \qquad \frac{d\vec{p}}{dt} = 0$$

$$\Rightarrow \vec{p} = \tau\vec{F} \qquad \Rightarrow \vec{v} = \frac{\tau}{m}\vec{F}$$

For the Hall geometry $\vec{F} = -e\vec{E} - e\frac{\vec{v}}{c} \times \vec{H}$ so

$$\vec{v} = -\frac{e\tau}{m}\vec{E} - \frac{e\tau}{mc}\vec{v} \times \vec{H}$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -\frac{e\tau}{m}\begin{pmatrix} E_x \\ E_y \end{pmatrix} - \frac{eH}{mc}\tau\begin{pmatrix} v_y \\ -v_x \end{pmatrix}$$

$\omega_c \equiv \dfrac{eH}{mc}$

$$\begin{pmatrix} v_x + \omega_c\tau v_y \\ v_y - \omega_c\tau v_x \end{pmatrix} = -\frac{e\tau}{m}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\frac{-m}{e\tau}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

current $\vec{j} = -en\vec{v}$ so $\vec{v} = -\dfrac{\vec{j}}{en}$

$$\frac{m}{me^2\tau}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\Rightarrow \overleftrightarrow{\rho} = \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} \qquad \text{with } \sigma_0 = \frac{ne^2\tau}{m}$$

ordinary dc conductivity

$$\Rightarrow \overleftrightarrow{\sigma} = \overleftrightarrow{\rho}^{-1} = \frac{\sigma_0}{1+(\omega_c\tau)^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}$$

$\Big($ This is same result as found in HW 1 for problem #1 if only look at one type of charge carrier, or problem #2 if set $\omega \to 0$. $\Big)$

We can relate the Hall coefficient R_H to the resistivity tensor as follows:

For a geometry in which the current flows along $\hat{x}$ then $j_y = 0$ and we define $R_H = \dfrac{E_y}{j_x H}$

$$j_y = 0 \Rightarrow \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}\begin{pmatrix} j_x \\ 0 \end{pmatrix} = \begin{pmatrix} \rho_{xx}\, j_x \\ -\rho_{xy}\, j_x \end{pmatrix}$$

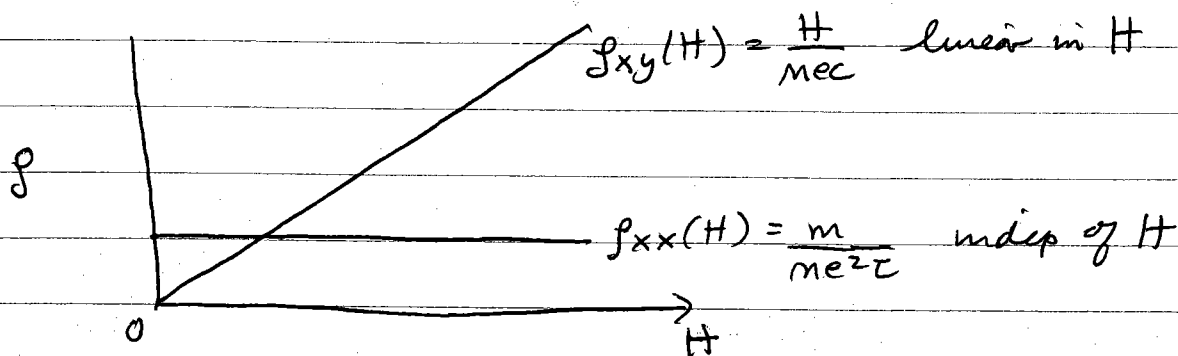so $R_H = \dfrac{E_y}{j_x H} = \dfrac{-\rho_{xy}\, j_x}{j_x H} = \boxed{\dfrac{-\rho_{xy}}{H} = R_H}$

R_H is just the off-diagonal element of $\overleftrightarrow{\rho}$ divided by H

From above:

$$\begin{cases} \rho_{xx} = \dfrac{1}{\sigma_0} = \dfrac{m}{me^2\tau} \\[2em] \rho_{xy} = \dfrac{\omega_c \tau}{\sigma_0} = \dfrac{eH}{mc}\dfrac{\tau}{me^2\tau}\,m = \dfrac{H}{mec} \end{cases}$$

(compare with $R_H = -\dfrac{\rho_{xy}}{H} = -\dfrac{1}{mec}$ we get back our earlier result for R_H)

So classically we expect

$\rho_{xy}(H) = \dfrac{H}{mec}$ linear in H

$\rho_{xx}(H) = \dfrac{m}{me^2\tau}$ indep of H

Note that in the limit of no electron scattering, ie $\tau \to \infty$, then $\rho_{xx} = \sigma_{xx} = 0$, $\rho_{xy} = \dfrac{H}{mec}$, we have a dissipationless state with $\vec{\jmath} \perp \vec{E}$.

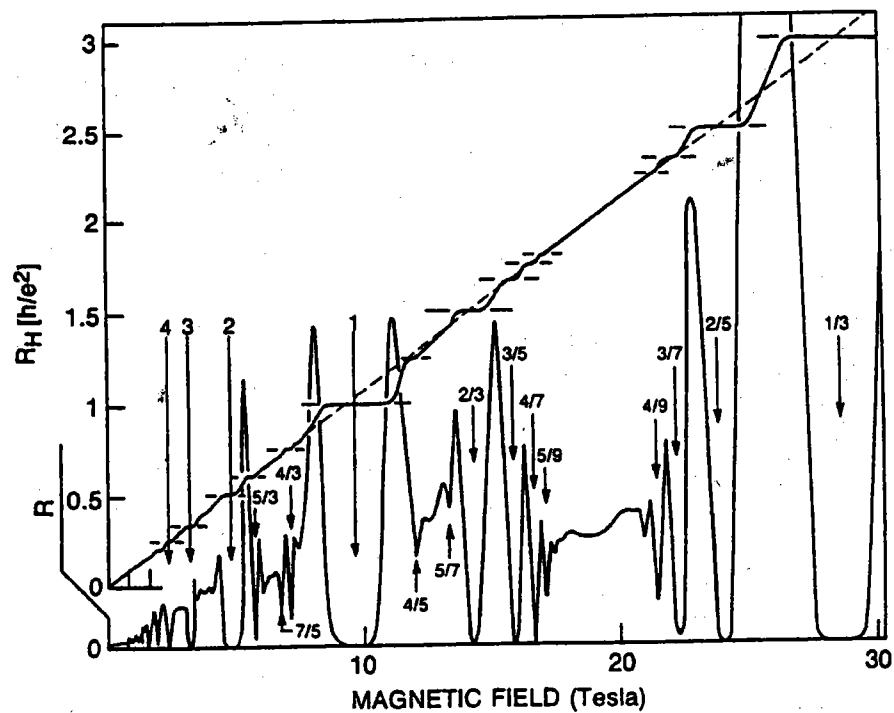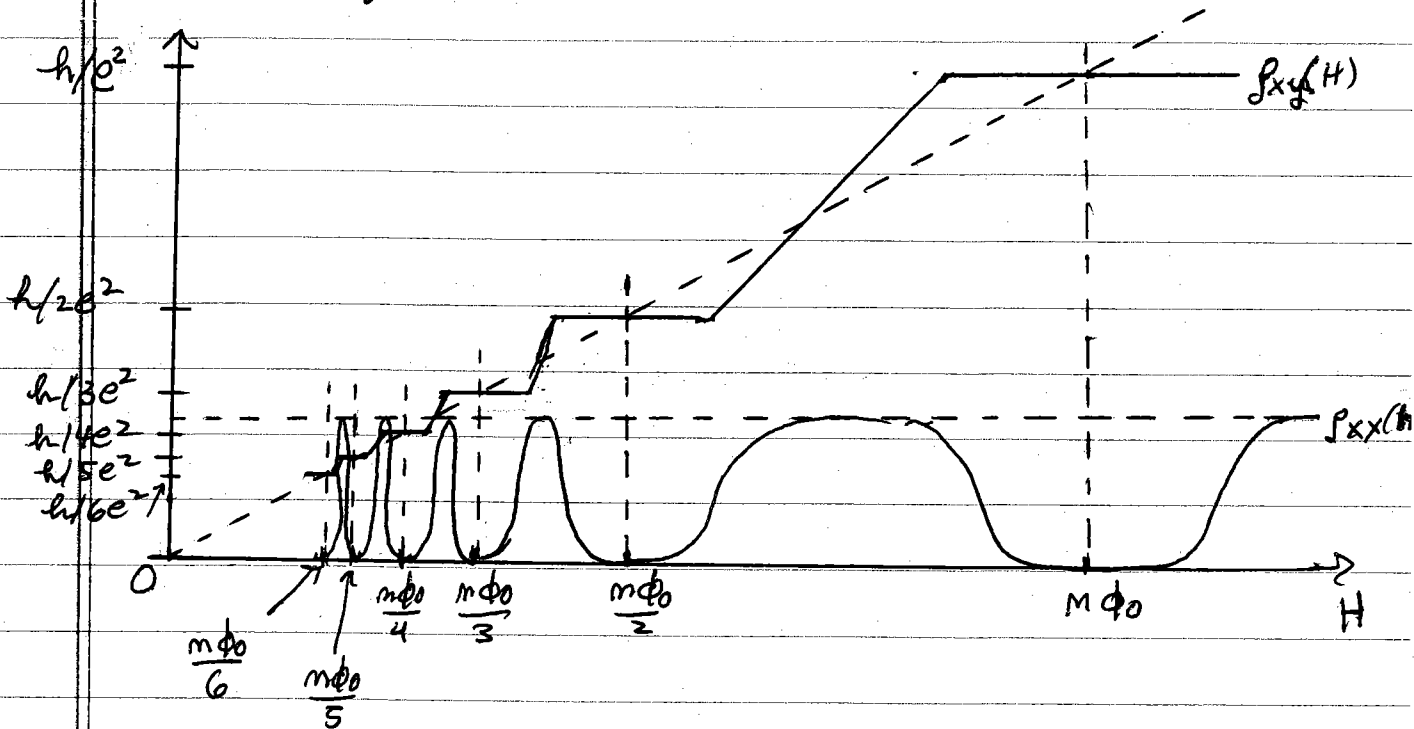*Experimental results showing both the integer and the fractional quantum Hall effect.*

Figure 1.2: Integer and fractional quantum Hall transport data showing the plateau regions in the Hall resistance R_H and associated dips in the dissipative resistance R. The numbers indicate the Landau level filling factors at which various features occur. After ref. [15].

Now compare to what one sees experimentally for a
two dimensional electron gas. An idealized sketch,
showing only the integer quantum Hall effect
looks as follows:

① $\rho_{xy}(H)$ consists of a series of plateaus at values
$\rho_{xy} = h/s e^2$ where s is an integer
the width of the plateaus increase as T decreases

② The plateaus are centered at the values of H which would
give $\rho_{xy} = h/s e^2$ in the classical Drude mode, ie.

$$\rho_{xy} = \frac{h}{s e^2} = \frac{H}{n e c} \Rightarrow H = \frac{m}{s} \frac{hc}{e} = \frac{m}{s} \phi_0 \quad \text{with } \phi_0 \equiv \frac{hc}{e}$$

Recall our earlier discussion of Landau levels where
we defined the flux quantum $\bar{\Phi}_o = \frac{hc}{2e}$. This new
ϕ_0 therefore has dimension of magnetic flux, and $\phi_0 = 2 \bar{\Phi}_o$.

$\frac{H}{\Phi_0}$ was the degeneracy of a Landau level if we counted electrons with both spin up and spin down and ignored the coupling of electron spin to H.

$\frac{H}{\Phi_0}$ is then the degeneracy of a Landau level if we consider only electrons of one fixed spin orientation (ie all up or all down).

③ In the center of the plateaus, for some interval of H about $H = m\Phi_0/a$, we have

$$\rho_{xx} = 0$$

The IQHE states at $H = \frac{m\Phi_0}{a}$ are therefore dissipationless current carrying states with quantized off diagonal conductivity,

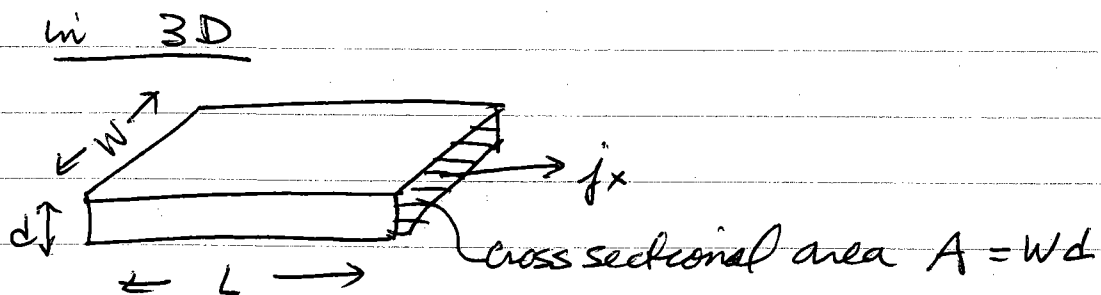$$-\sigma_{xy} = \frac{1}{\rho_{xy}} = a \cdot \frac{e^2}{h}$$

$\frac{h}{e^2}$ has the units of conductance and is called the "quantum of conductance".

Because σ_{xy} is quantized in integer units of h/e^2, the quantum Hall effect is now used as a standard for measuring conductance

Why two dimensions is important

In a theoretical model, it is resistively ρ_{xx} and ρ_{xy} that one computes. These are the quantities that are intrinsic to the material and the physical situation. The total Resistance R_{xx}, R_{xy} are related to ρ_{xx} and ρ_{xy} by sample geometry. However, while theory calculates ρ, it is R that is directly measured in experiment. So to convert from measured R to prediction for ρ it would seem that one needs to know very precisely the correct geometrical factors for this conversion.

Not true in two dimensions.

This is important because the theory predicts that ρ_{xy} is quantized in units of h/se^2, a integer, whereas it is R_{xy} that is the directly measured quantity. We want to be able to test the theory with out worrying about the geometry! To illustrate this, consider a simple rectangular geometry

in 3D

cross sectional area $A = Wd$

for geometry in which current flows along $\hat{x}$ and so $j_y = 0$, we have

but

$$R_{xy} = \frac{V_y}{I_x} = \frac{W E_y}{W j_x} = \frac{E_y}{j_x} = \rho_{xy}$$

So $\boxed{R_{xy} = \rho_{xy}}$ in 2D

R_{xy} and ρ have the same units in 2D and there is no geometrical factor relating R_{xy} to ρ_{xy}! Direct measurement of R_{xy} gives the quantized ρ_{xy}.

Note the units of $\rho_{xy} = h/se^2$ in 2D have dimensions of $\frac{h}{e^2} = \frac{energy \cdot sec}{energy \cdot length} = \frac{sec}{length}$

while the units of R in 2D have dimensions of

$$R = \frac{V}{I} = \frac{energy/charge}{charge/sec} = \frac{energy \cdot sec}{(charge)^2}$$

$$= \frac{energy \cdot sec}{energy \cdot length} = \frac{sec}{length}$$

So $\frac{h}{e^2}$ has units of total resistance = units of resistivity only in two dimensions.

The conductance tensor $\overleftrightarrow{G} = \overleftrightarrow{R}^{-1}$ inverse of resistance tensor. In quantum Hall state with $\rho_{xx} = \sigma_{xx} = 0$,
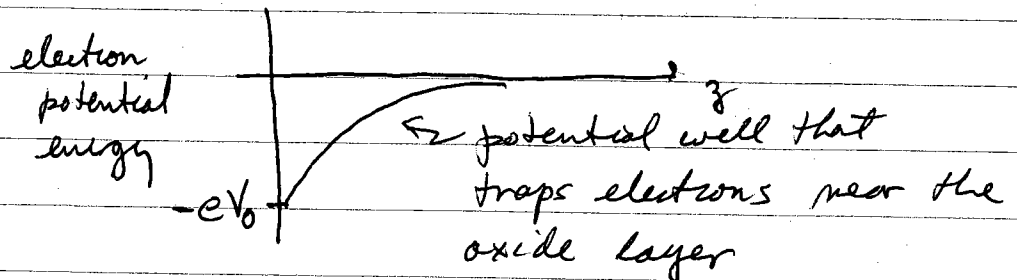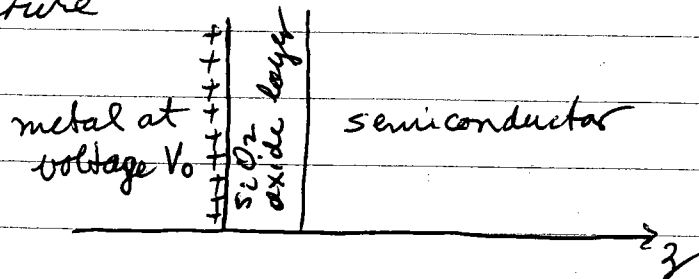
$$-G_{xy} = \frac{1}{R_{xy}} = \frac{1}{\rho_{xy}} = -\sigma_{xy}$$

$G_{xy} = \sigma_{xy}$ Conductance and conductivity have the same units, and transverse conductance = transverse conductivity

Aside: How does one make a two dimensional electron gas experimentally?

Semiconductor inversion layer

simple picture

metal at voltage V_0 — Si O$_2$ oxide layer — semiconductor

z

electron potential energy

$-eV_0$

← potential well that traps electrons near the oxide layer

Adjust V_0 so that the width of the potential well $\approx$ de Broglie wavelength of the electrons in the semiconductor $\Rightarrow$ motion in $\hat{z}$ direction is quantized into discrete levels. For proper choice of V_0, and low T, one can insure that all electrons in the ground state lie in the lowest z level. Electrons remain free in x and y directions parallel to the interface,

When one applies $\vec{H} = H\hat{z}$, the energy levels are given by the Landau levels
$$E_n = \hbar \omega_c (n + 1/2) + \text{const}$$

↑
lowest eigenvalue
for motion along z

Questions

(1) When $H = \dfrac{n \phi_0}{a}$, why is $f_{xy} = \dfrac{h}{n e^2}$ the classical Drude value, and why is $f_{xx} = 0$?

(2) Why is $f_{xy} = \dfrac{h}{n e^2}$ constant for a range of H about $H = \dfrac{n \phi_0}{a}$, ie why are there the flat plateaus in f_{xy}?

Partial answer to (1)

Consider the Landau level structure of the eigenstates of electron motion in the xy plane. We need now to also consider the interaction of intrinsic electron spin with the magnetic field.

For an electron in Landau level n with spin $s = \pm 1$, the energy is

$$\varepsilon(n, s) = \hbar \omega_c \left(n + \tfrac{1}{2}\right) \pm \mu_0 H$$

with $\mu_0 = \dfrac{e \hbar}{2 m c}$ the Bohr magneton

Now note $\mu_0 H = \dfrac{e H}{m c} \dfrac{\hbar}{2} = \tfrac{1}{2} \hbar \omega_c$

so
$$\varepsilon(n,s) = \begin{cases} \hbar\omega_c(n+1) & \text{for } s = +1 \\ \hbar\omega_c\, n & \text{for } s = -1 \end{cases}$$
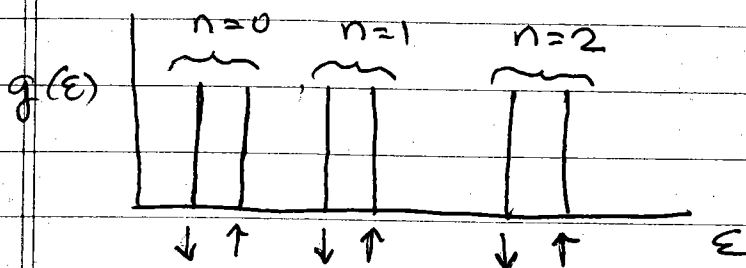
The up electron energy of Landau level n coincides with the down electron energy of Landau level $n+1$. The degeneracy of each Landau level remains $\dfrac{H}{\Phi_0}$ with $\Phi_0 = \dfrac{hc}{2e}$.

In practice, for electrons in a real metal or semiconductor, the "mass" that enters their equation of motion is not the free electron mass m, but rather some effective mass m^* that includes effects of the periodic ionic potential in which the electron moves. It is this m^* that enters the ω_c of the energy of the Landau levels, $\omega_c = \dfrac{eH}{m^*c}$. However, for the interaction of the intrinsic spin with H, the Bohr magneton involves the free electron mass m. Hence there is no overlap of $n\!\uparrow$ with $(n+1)\!\downarrow$ energy levels as there would be for a true free electron gas, ie now

$$e(n,s) = \frac{\hbar eH}{m^*c}\left(n+\tfrac{1}{2}\right) \pm \frac{e\hbar}{2mc}H$$

In general $m^* \neq m$ so the Landau level for $S = +1$ is split off from the Landau level for $S = -1$, with no overlap between ~~them~~ them

The density of states then looks like

Now, since the spin states of each Landau level are split off from each other, the degeneracy of each δ-function in the above $g(\varepsilon)$ is half what it was when we ignore coupling of spin to H.

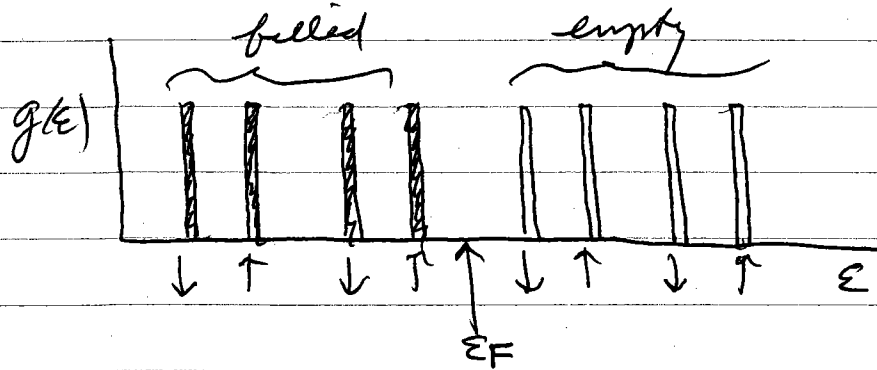$\Rightarrow$ degeneracy of each Landau level is now

$$\frac{H}{2\Phi_0} = \frac{H}{\Phi_0} \qquad \text{with } \phi_0 = \frac{hc}{e}$$

<u>Return to partial answer to question ①</u>

when $H = \frac{m \phi_0}{\Delta}$ we have $m = \Delta \frac{H}{\Phi_0}$

$\Rightarrow$ we have exactly Δ lowest Landau levels completely filled, and all higher Landau levels completely empty (Δ is integer)

There is thus an *finite energy gap* to the
lowest unoccupied single particle energy level

when $\mu_0 H, \hbar \omega_c \gg k_B T$

and $\mu_0 H, \hbar \omega_c \gg$ other energy scales that
describe collisions, such as
$\hbar \omega_D$, where ω_D is the max
phonon freq, or $e E v_F \tau$, the
max energy absorbed by electron
from $\vec{E}$ field between collisions

Then electrons in filled Landau levels cannot
scatter into an unoccupied state in an empty
Landau level. There are no empty states
close enough in energy.

When electrons cannot scatter, $\tau \to \infty$,
we have $\sigma_{xx} = \rho_{xx} = 0$ dissipationless
current flow.

This explains why $\rho_{xx} = 0$ when $H = \dfrac{n \phi_0}{s}$
for integer s!