Motion in uniform $\perp$ $\vec{E}$ and $\vec{H}$ fields
Hall effect and magnetoresistence

$$\hbar \dot{\vec{k}} = -e\left[\vec{E} + \frac{\vec{v}}{c}(\vec{k}) \times \vec{H}\right]$$

$$\Rightarrow \quad \hat{H} \times \hbar \dot{\vec{k}} = -e\,\hat{H} \times \vec{E} - \frac{eH}{c}\dot{\vec{r}}_\perp$$

$$\dot{\vec{r}}_\perp = -\frac{\hbar c}{eH}\hat{H} \times \dot{\vec{k}} + \vec{w} \qquad \vec{w} = \frac{cE}{H}(\hat{E} \times \hat{H})$$

Motion is as before, but with drift velocity $\vec{w}$ added.

To determine orbits in k space note:

$$\hbar \dot{\vec{k}} = -e\vec{E} - \frac{e}{c}\frac{1}{\hbar}\frac{\partial \mathcal{E}}{\partial \vec{k}} \times \vec{H} \qquad \text{write } \vec{E} = -(\vec{E} \times \hat{H}) \times \hat{H}$$

true when $\vec{E} \perp \vec{H}$

$$= -\frac{e}{c\hbar}\left(\frac{\partial \mathcal{E}}{\partial \vec{k}} - \frac{c\hbar E}{H}\hat{E} \times \hat{H}\right) \times \vec{H}$$

$$\equiv -\frac{e}{c\hbar}\frac{\partial \bar{\mathcal{E}}}{\partial \vec{k}} \times \vec{H} \qquad \bar{\mathcal{E}} = \mathcal{E} - \hbar\vec{k} \cdot \vec{w}$$

Same as if $\vec{E}$ was absent and band structure replaced by

$$\bar{\mathcal{E}}(\vec{k}) = \mathcal{E}(\vec{k}) - \hbar\vec{k} \cdot \vec{w}$$

Orbits are intersections of surfaces of constant $\bar{\mathcal{E}}$ with planes $\perp$ to $\vec{H}$

We will assume that $-\hbar\vec{k} \cdot \vec{w}$ small enough so that if the constant $\mathcal{E}(k)$ surface is closed (open) so is the constant $\bar{\mathcal{E}}(k)$ surface. Good approx in most cases - see text for estimate of numbers.

in nearly free electron model

$$\varepsilon(\vec{k}) \simeq \frac{\hbar^2}{2m}\vec{k}^2$$

surface of constant energy ε is sphere of radius $\sqrt{\frac{2m\varepsilon}{\hbar^2}} = k$ in k-space

$$\bar{\varepsilon}(\vec{k}) = \frac{\hbar^2\vec{k}^2}{2m} - \hbar\vec{w}\cdot\vec{k}$$

surface of constant $\bar{\varepsilon}$ is given by

$$\frac{\hbar^2}{2m}\left|\vec{k} - \frac{m\vec{w}}{\hbar}\right|^2 = \bar{\varepsilon} + \tfrac{1}{2}mw^2$$

sphere in k-space of radius

$$k = \sqrt{\frac{2m}{\hbar^2}\left(\bar{\varepsilon} + \tfrac{1}{2}mw^2\right)}$$

centered about $\vec{k}_0 = m\vec{w}/\hbar$

surface of constant $\bar{\varepsilon}$ is shifted by $\vec{w}\cdot\vec{k}$ term in direction $\vec{w}$

Hall effect: $\dot{\vec{r}}_\perp = -\frac{\hbar c}{eH}\hat{H}\times\dot{\vec{k}} + \vec{w}$, $\vec{w} = \frac{cE}{H}(\hat{E}\times\hat{H})$

current in plane $\perp$ to H is

$\vec{j} = -ne\langle\dot{\vec{r}}_\perp\rangle$ where $\langle\dot{\vec{r}}_\perp\rangle$ is steady state average over all occupied electron orbits, and over collisions

$\vec{j} = -ne\vec{w} + \frac{ne\hbar c}{eH}\hat{H}\times\langle\dot{\vec{k}}\rangle$

Case (1) All occupied (or unoccupied) orbits are closed. Then for large enough H so that $\omega_c\tau \gg 1$ (where τ is collision time, and $\omega_c = eH/m^*c$), electron makes many periods of its closed orbits between sucessive collisions.

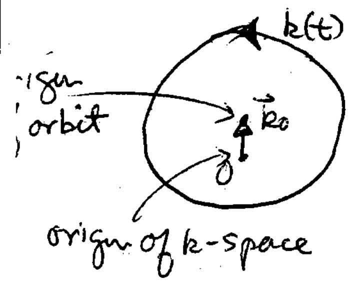
We can estimate $\langle\dot{\vec{k}}\rangle$ in this large H case as follows: Averaging over electron motion between two sucessive collisions at $t=0$ and $t=t_0$ we get

$$\langle\dot{\vec{k}}\rangle = \frac{1}{t_0}\int_0^{t_0}\dot{\vec{k}}(t)\,dt = \frac{\vec{k}(t_0)-\vec{k}(0)}{t_0}$$

where $\vec{k}(0)$ is wave vector of electron as it emerges from the first collision at $t=0$, and $\vec{k}(t_0)$ is wave vector of electron just before second collision at $t=t_0$

As in the Drude model, we may assume that electrons emerge from a collision with an equilibrium distribution determined by the local temperature + chemical potential. Since the Fermi distribution $f(\varepsilon(k)) = \frac{1}{1+e^{\beta(\varepsilon-\mu)}}$ depends on $\vec{k}$ only via energy $\varepsilon(\vec{k})$, and $\varepsilon(\vec{k}) = \varepsilon(-\vec{k})$, we have, after averaging over the electron energy from the collision at $t=0$, $\langle \vec{k}(0)\rangle = 0$. So $\langle \dot{\vec{k}}\rangle = \vec{k}(t_0)/t_0$.

We now average over the time until the second collision, $\langle t_0\rangle = \tau$ (this time is distributed randomly with average equal to τ). Since $\omega_c\tau \gg 1$, the electron makes *many* orbits between collisions, $\Rightarrow \vec{k}(t_0)$ when averaged over collision time t_0, is equally likely to lie anywhere along the closed orbit.
$\Rightarrow \langle \vec{k}(t_0)\rangle$ = (average $\vec{k}$ on orbit). If electric field $\vec{E}=0$, then (average $\vec{k}$ on orbit) $=0$ also. But when $E \neq 0$, (average $\vec{k}$ on orbit) $\sim m^*\vec{w}/\hbar$. To see this, use effective mass approximation, $\varepsilon(k) \simeq \dfrac{\hbar^2 k^2}{2m^*}$.
Then orbit lies on curve of constant
$\bar{\varepsilon}(k) = \varepsilon(k) - \hbar\vec{k}\cdot\vec{w}$, which lies on sphere centered at $\vec{k}_0 = m^*\vec{w}/\hbar$. So (average $\vec{k}$ on orbit) $= \langle \vec{k}(t_0)\rangle = \vec{k}_0$



$$\Rightarrow \langle \dot{\vec{k}}\rangle = \frac{\langle \vec{k}(t_0)\rangle}{\tau} = \frac{\vec{k}_0}{\tau} = \frac{m^*\vec{w}}{\hbar\tau}.$$

So contribution of $\langle \dot{k}\rangle$ term to current is

$$\frac{ne\hbar c}{eH}\,\hat{H}\times\frac{m^*\vec{w}}{\hbar\tau} = \frac{ne}{\omega_c\tau}\,\hat{H}\times\vec{w}$$

smaller than drift contribution to current $\vec{j} \simeq -ne\vec{w}$ by a factor $\dfrac{1}{\omega_c\tau} \ll 1$

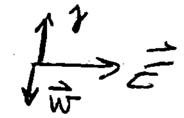
So $\vec{j} \simeq -ne\vec{w}$ given just by drift velocity $\vec{w}$ in high field limit.

In this case $\vec{j}$ is $\parallel$ to $\vec{w}$ $\Rightarrow$ $\vec{j}$ is $\perp$ to $\vec{E}$ and $\vec{H}$
$\Rightarrow$ Lorentz force so strong that electrons move $\perp$ to E and do not acquire any energy from the E-field.

The Hall coefficient in this limit is just

$$R_{H\to\infty} = \frac{E_\perp}{jH}$$

E-field perpendicular to H (and $\perp$ to $\vec{j}$), but in this large H limit, this is just total E.



$$= \frac{E}{-new\,H}$$

but $\vec{w} = \frac{cE}{H}(\hat{E}\times\hat{H})$

$$\Rightarrow R_{H\to\infty} = \frac{E}{-ne\frac{cE}{H}H} = \frac{-1}{nec}$$

Drude value

The above was for <u>closed occupied orbits</u>
If we had closed <u>unoccupied orbits</u> we would use the hole picture to get

$$R_{H\to\infty} = +\frac{1}{n_h ec} > 0$$

(n_h is density of holes, each hole has charge $+e$)

If there is more than one partically full band with only closed occupied or unoccupied orbits, then

$$\vec{j} = -n_{eff}\frac{ec}{H}(\vec{E}\times\hat{H}) \quad \text{where } n_{eff} = n - n_h$$

n: total electron density in all partially full bands

n_h: total hole density in partially full bands

$$R_{H\to\infty} = \frac{-1}{n_{eff}\,ec}$$

The effects of holes explains why R_∞ can have non Drude values, and even be >0.

See text for what happens when $m_{eff} = 0$. This is case for undoped semiconductor

Another way to view things is ~~not~~ in terms of conductivity tensor. keeping contribution to $\vec{j}$ from the $\langle \dot{\vec{k}} \rangle$ term gives

$$\vec{j} = -ne\vec{w} + \frac{ne}{\omega_c\tau}\hat{H} \times \vec{w} \quad , \quad \vec{w} = \frac{cE}{H}(\hat{E} \times \hat{H})$$

for $\hat{H} = \hat{z}$ direction we have

$$\vec{j} = \frac{nec}{H}\left(\hat{z} \times \vec{E} + \frac{1}{\omega_c\tau}\vec{E}\right) = \overleftrightarrow{\sigma} \cdot \vec{E}$$

with $\overleftrightarrow{\sigma} = \frac{nec}{H}\begin{pmatrix} \frac{1}{\omega_c\tau} & -1 \\ 1 & \frac{1}{\omega_c\tau} \end{pmatrix}$

or writing $\frac{\sigma_0}{\omega_c\tau} = \frac{ne^2\tau}{m^*}\frac{m^*c}{eH\tau} = \frac{nec}{H}$ where

σ_0 is Drude conductivity $\Rightarrow \overleftrightarrow{\sigma} = \sigma_0 \begin{pmatrix} \frac{1}{(\omega_c\tau)^2} & -\frac{1}{\omega_c\tau} \\ \frac{1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} \end{pmatrix}$

(Compare with prob #1 on HW #1 !!)

$\Rightarrow$ resistivity tensor $\overleftrightarrow{\rho} = \overleftrightarrow{\sigma}^{-1} = \dfrac{1/\sigma_0}{\frac{1}{(\omega_c\tau)^4} + \frac{1}{(\omega_c\tau)^2}} \begin{pmatrix} \frac{1}{(\omega_c\tau)^2} & +\frac{1}{\omega_c\tau} \\ \frac{-1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} \end{pmatrix}$

$\Rightarrow$ ~~Hall coefficient is $R = -\frac{\rho_{xy}}{H}$~~

as $\left(\frac{1}{\omega_c\tau}\right) \ll 1$

$$\overleftrightarrow{\rho} = \frac{1/\sigma_0}{1 + \frac{1}{(\omega_c\tau)^2}}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} \simeq \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}$$

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E} \qquad \overleftrightarrow{\sigma} = \frac{\sigma_0}{\omega_c\tau}\begin{pmatrix} \frac{1}{\omega_c\tau} & -1 \\ 1 & \frac{1}{\omega_c\tau} \end{pmatrix}$$

$$\sigma_0 = \frac{ne^2\tau}{m^*}$$

$$\omega_c\tau = \frac{eH\tau}{m^*c} \gg 1$$

Then $\vec{E} = \overleftrightarrow{\rho} \cdot \vec{j}$ where $\overleftrightarrow{\rho} \approx \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}$

For $\vec{j} = j\hat{x}$ then $E_y = \rho_{yx} j = -\rho_{xy} j$

Hall coef: $R = \frac{E_y}{jH} = -\frac{\rho_{xy}}{H} = \frac{-\omega_c\tau}{\sigma_0 H} = -\frac{eH\tau}{m^*c}\,\frac{m^*}{ne^2\tau}\,\frac{1}{H}$

$= -\frac{1}{nec}$ Drude value

~~For holes , $\vec{j} = n_h$~~

- For electrons we used $\vec{j} = -ne\vec{w} + \frac{ne}{\omega_c\tau}\vec{H}\times\vec{w}$

For holes we use instead $\vec{j} = +n_h e\vec{w} - \frac{n_h e}{\omega_c\tau}\vec{H}\times\vec{w}$
since charge carriers have charge $+e$.
All results carry through except take $e \to -e$

$$\Rightarrow R = \frac{1}{n_h ec}$$