

1) [25 points total]

Give a brief and to the point answer to each of the following questions. You may cite appropriate equations when it helps to explain a point, but no lengthy calculations should be necessary.

a) [6 points] Explain what is meant by a gauge transformation.

b) [6 points] A point charge q (assume $q > 0$) is positioned a large distance r from a neutral atom of atomic polarizability α . How does the force between the charge and the atom vary with distance r . Is the force repulsive or attractive? What happens if now $q < 0$?

c) [6 points] Explain three important consequences of having a complex, frequency dependent, dielectric function (as opposed to a real, frequency independent one).

d) [7 points] In the last homework set, you derived the Faraday effect. According to this effect a linearly polarized electromagnetic wave, traveling through a dielectric in the presence of uniform magnetic field, has its direction of polarization rotated as it travels down the dielectric. A naive application of the principle of superposition might argue against the existence of this effect: if a linearly polarized wave with fixed direction of polarization and uniform $\mathbf{B} = 0$ is one solution, and a uniform $\mathbf{B} \neq 0$ and no wave is another solution, then the sum of these is also a solution and so the presence of the uniform $\mathbf{B} \neq 0$ should in no way effect the propagation of the wave! What is wrong with this naive argument?

2) [25 points total]

Consider a dielectric sphere of radius R with dielectric constant ϵ . At the center of the sphere is a point charge q . The sphere is placed in a uniform external electric field \mathbf{E}_0 (that is, the field was uniform until the sphere was placed in it).

a) [17 points]

What is the electric field inside and outside the sphere?

b) [8 points]

What is the total surface charge density induced on the surface of the sphere?

3) [25 points total]

This problem may seem long, but its goal is to lead you step by step to the interesting conclusion that the energy absorbed by a medium from a propagating EM wave is determined by the imaginary part of the dielectric function.

Consider a bound atomic electron in a dielectric medium of polarizability $\alpha(\omega)$. The dielectric function is $\epsilon(\omega) = 1 + 4\pi N\alpha(\omega)$ and $\mu=1$ (N is the density of polarizable atoms).

An electric field $\mathbf{E}(t) = \text{Re}[\mathbf{E}_\omega e^{i(\delta - \omega t)}]$ exerts a force on the electron of charge q . Here \mathbf{E}_ω is a real valued vector, and δ is an arbitrary phase factor.

a) [6 points] Show that the velocity of the electron is given by

$$\mathbf{v}(t) = \text{Re} \left[\frac{-i\omega}{q} \alpha(\omega) \mathbf{E}_\omega e^{i(\delta - \omega t)} \right]$$

b) [7 points] Show that the work done on the electron by the electric field, per one period of oscillation (this is just the average power absorbed by the electron), is

$$\frac{1}{2} \omega \alpha_2(\omega) |\mathbf{E}_\omega|^2$$

where $\alpha_2(\omega) = \text{Im}[\alpha(\omega)]$. (Hint: Be careful to work with the real valued forms, rather than the complex exponential expressions - you always have to be careful about this in taking the product of two such oscillating quantities.)

c) [6 points] Consider an electromagnetic wave traveling in the $+z$ direction. In lecture we showed that the electric field of such a wave had the form

$$\mathbf{E}(z, t) = \text{Re} \left[\mathbf{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)} \right] = \mathbf{E}_\omega e^{-k_2 z} \cos(k_1 z - \omega t)$$

where $k = k_1 + ik_2$ is the complex wave vector, determined from the complex

dielectric permeability $\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$ by the expression $k^2 = \frac{\omega^2}{c^2} \epsilon(\omega)$.

Assume the material has a cross-sectional area A , in the xy plane, and is semi-infinite in the z direction from $z=0$ to $z \rightarrow \infty$. Using the form above, and the result in part (b), compute the total work per unit cross-sectional area, per period of oscillation, done by the electromagnetic wave on the material. This is the total energy per unit area, per period of oscillation, lost by the wave in traveling through the material. Show that it is equal to

$$\frac{1}{16\pi} \frac{\omega}{k_2} \epsilon_2(\omega) |\mathbf{E}_\omega|^2$$

d) [6 points] Show that for a frequency ω in the region of total reflection the work done on the material is zero.

4) [25 points total]

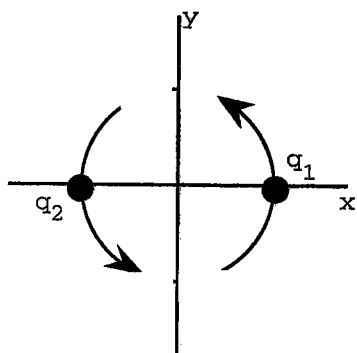
For a time dependent charge distribution with an oscillating electric dipole moment given by $\mathbf{p}(t) = \text{Re}[\mathbf{p}_\omega e^{-i\omega t}]$, the radiated fields in the electric dipole approximation are given by,

$$\mathbf{E}(\mathbf{r}, t) = \text{Re} \left[-k^2 \frac{e^{i(kr - \omega t)}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega) \right],$$

$$\mathbf{B}(\mathbf{r}, t) = \text{Re} \left[k^2 \frac{e^{i(kr - \omega t)}}{r} \hat{\mathbf{r}} \times \mathbf{p}_\omega \right],$$

where $k = \omega/c$.

Consider two charges, q_1 and q_2 , which are moving in a circular orbit of radius d with angular frequency ω , in the xy plane, as shown below.



position of q_1 is: $d(\hat{x} \cos \omega t + \hat{y} \sin \omega t)$

position of q_2 is: $-d(\hat{x} \cos \omega t + \hat{y} \sin \omega t)$

a) [15 points]

If $q_1 = -q_2 = q$, find the radiated \mathbf{E} and \mathbf{B} fields in the electric dipole approximation. Express your answers as REAL functions of space and time. Find the time averaged Poynting vector as a function of space. Make a polar plot of the angular distribution of the radiated power $dP/d\Omega$.

(Hint: The key to this problem is realizing how to write the amplitude \mathbf{p}_ω correctly. Note: the angular distribution here will NOT have the simple $\sin^2\theta$ form we found in lecture for electric dipole radiation - think about what is different to see why.)

b) [10 points]

What happens if one now has the case $q_1 = q_2 = q$? What term in the multipole expansion is responsible for the radiation? What is the frequency of the radiation? (You do not need to explicitly calculate \mathbf{E} , \mathbf{B} or \mathbf{S} .)