Magnetostatics

$$\vec{\nabla}\cdot\vec{B} = 0$$

$\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$ Ampere's law (statics only!)

$$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{j}$$

can write $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

where by $\nabla^2\vec{A}$ we mean $(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$

$\nabla^2\vec{A}$ only has a <u>single</u> expression in <u>Cartesian coords</u>

Aside

If tried to write it in spherical coords, for example, one has

$$\begin{aligned}\nabla^2\vec{A} &= \nabla^2(A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}) \\ &= (\nabla^2 A_r)\hat{r} + A_r(\nabla^2\hat{r}) + (\nabla^2 A_\theta)\hat{\theta} + A_\theta(\nabla^2\hat{\theta}) \\ &\quad + (\nabla^2 A_\varphi)\hat{\varphi} + A_\varphi(\nabla^2\hat{\varphi})\end{aligned}$$

one must not forget to take the derivates of $\hat{r}, \hat{\theta}, \hat{\varphi}$ since they vary with position!

for example, $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

one could compute $\nabla^2\hat{r}$ by applying ∇^2 in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla}\cdot\vec{A} = 0$, then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \boxed{-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}}$$

Poisson's equation!

Many of the same methods used to solve for electrostatic ϕ can therefore be applied to solve for magnetostatic $\vec{A}$. But vector nature of eqn makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ three equations for A_x, A_y, A_z !

for localized current sources $j(r) \to 0$ as $r \to \infty$

Multipole expansion - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use vector spherical harmonics - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{1/2}} = \frac{1}{r}\,\frac{1}{\left[1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

do Taylor series to 1st order in $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r}\left\{1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \cdots\right\} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots$$

$$\vec{A}(\vec{r}) = \underbrace{\int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}\,')}{r}}_{①} + \underbrace{\int \frac{d^3r'}{c} \vec{j}(\vec{r}\,') \frac{(\vec{r}\cdot\vec{r}\,')}{r^3}}_{②} + \cdots$$

Consider term ①

$$\int d^3r \, \vec{j}(\vec{r})$$

$\frac{\partial r_i}{\partial r_j} = \delta_{ij}$

write $\int d^3r \, j_i(r) = \sum_{j=1}^{3} \int d^3r \, j_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

$$= \sum_j \left\{ \oint_S da \, j_j \, r_i - \int d^3r \frac{\partial j_j}{\partial r_j} r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized ie $\vec{j}(\vec{r}) \to 0$ sufficiently fast as $r \to \infty$

vanishes in magnetostatics where $\vec{\nabla}\cdot\vec{j} = 0$

So $\int d^3r \, \vec{j}(\vec{r}) = 0$ in magnetostatics
monopole term vanishes

term (2)

$$\int d^3r \; \vec{j}(\vec{r})\, \vec{r} \qquad \text{tensor}$$

Consider $\int d^3r \; j_i r_j = \sum_k \int d^3r \; j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts

$$= \sum_k \left\{ \oint_S da \; j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k} (j_k r_j)\, r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized

$$= -\sum_k \int d^3r \left(\frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

vanishes as $\vec{\nabla}\cdot\vec{j} = 0$ in magnetostatics $\qquad = \delta_{jk}$

$$= -\int d^3r \; j_j r_i$$

So $\int d^3r \; j_i r_j = -\int d^3r \; j_j r_i$

$$= \frac{1}{2} \int d^3r \left(j_i r_j - j_j r_i \right)$$

So

$$\int d^3r' \; j_i(\vec{r}')(\vec{r}\cdot\vec{r}') = \sum_j r_j \int d^3r' \, j_i(\vec{r}')\, r_j'$$

$$= \sum_j \frac{1}{2} \int d^3r' \left(j_i r_j r_j' - r_j j_j r_i' \right)$$

$$= \frac{1}{2} \int d^3r' \left(j_i (\vec{r}\cdot\vec{r}') - r_i' (\vec{r}\cdot\vec{j}) \right)$$

use triple product rule

$$\vec{r} \times (\vec{r}' \times \vec{J}) = \vec{r}' (\vec{r} \cdot \vec{J}) - \vec{J} (\vec{r} \cdot \vec{r}')$$

to rewrite as

$$\int d^3r' \, \vec{J} \, (\vec{r} \cdot \vec{r}') = -\frac{1}{2} \vec{r} \times \left[\int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}') \right]$$

define the magnetic dipole moment as

$$\boxed{\vec{m} = \frac{1}{2c} \int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}')}$$

In magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}(\vec{r}) = -\frac{\vec{r} \times \vec{m}}{r^3} = \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\vec{m} \times \hat{r}}{r^2}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3} \right)$$

$$B_i = \varepsilon_{ijk} \, \partial_j \, \varepsilon_{klm} \, m_l \frac{r_m}{r^3}$$

$$= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) \partial_j \, m_l \frac{r_m}{r^3}$$

$$= m_i \, \partial_j \left(\frac{r_j}{r^3} \right) - m_j \partial_j \left(\frac{r_i}{r^3} \right)$$

$$= m_i \left[-4\pi \delta(\vec{r}) \right] - m_j \left[\frac{\delta_{ij}}{r^3} - \frac{3 r_i}{r^4} \partial_j r \right]$$

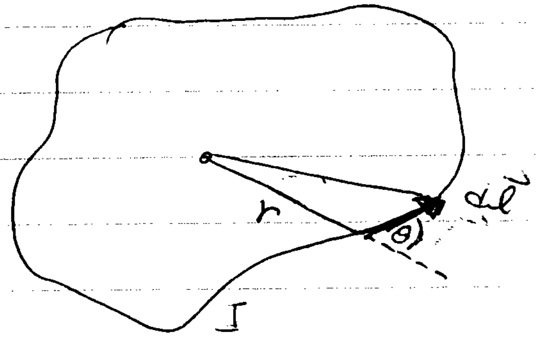
$$= \underset{\text{far from source}}{0} - \frac{m_i}{r^3} + 3 \frac{r_i}{r^4} \frac{r_j}{r} m_j$$

$$\boxed{\vec{B} = \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}}$$

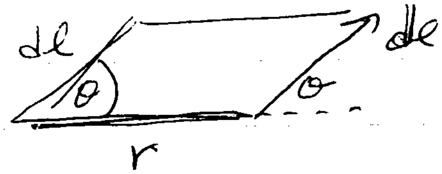
same form as $\vec{E}$ from electric dipole $\vec{p}$

For a <u>current loop in a plane</u> (any shape loop provided it is flat)

$$\vec{m} = \frac{1}{2c}\int d^3r\ \vec{r}\times\vec{j} = \frac{1}{2c} I \oint \vec{r}\times d\vec{l}$$



area of triangle is $\frac{1}{2} r\, dl \sin\theta$
$= \frac{1}{2} |\vec{r}\times d\vec{l}|$



area of rectangle is $r\, dl \sin\theta$

$$\Rightarrow \quad \vec{m} = \frac{1}{c} I (\text{area})\, \hat{m}$$

area of loop

outward normal (direction given by right hand rule with respect to direction of current) magnetic dipole moment $\vec{m}$ is independent of location of origin.

$$\vec{r}\,' = \vec{r} + \vec{d} \qquad \text{new coord}$$

$$\vec{m}\,' = \frac{1}{2c}\int d^3r'\,(\vec{r}\,'\times\vec{j}) = \frac{1}{2c}\int d^3r\,(\vec{r}+\vec{d})\times\vec{j}$$

$$= \frac{1}{2c}\int d^3r\,\vec{r}\times\vec{j} + \frac{1}{2c}\vec{d}\times\left[\int d^3r\,\vec{j}\,\right]$$

$$\vec{m}\,' = \vec{m} + 0 \quad \text{as} \quad \int d^3r\,\vec{j} = 0$$

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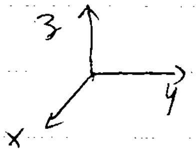
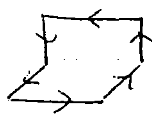
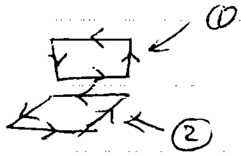
for planar loop $\vec{m} = \frac{I}{c} a\,\hat{n}$ where a = area, $\hat{n}$ = outward normal

Can also apply to get $\vec{m}$ for <u>piecewise planar loops</u>





= superposition of



$$\vec{m} = \vec{m}_1 + \vec{m}_2 \qquad \vec{m}_1 = \frac{I}{c} a_1\,\hat{x}$$
$$\vec{m}_2 = \frac{I}{c} a_2\,\hat{z}$$

$$\Rightarrow \quad \vec{m} = \frac{I}{c}(a_1\hat{x} + a_2\hat{z})$$