when atoms have intrinsic magnetic moments due to electron spin, we can add these to $\vec{M}$ in obvious way

when molecules are neutral, $q_n = 0$, the "bound current" is given by

$$\vec{J}_{bound} = \sum_n \langle \vec{J}_n \rangle = c\,\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Note that the $\frac{\partial \vec{P}}{\partial t}$ term is crucial to give conservation of bound charge

$$\vec{\nabla}\cdot\vec{J}_{bound} = c\,\vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) + \vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t}$$

$$= 0 + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P})$$

$$= -\frac{\partial \rho_{bound}}{\partial t}$$ where $\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$ is bound charge density

So $\boxed{\vec{\nabla}\cdot\vec{J}_{bound} + \frac{\partial \rho_{bound}}{\partial t} = 0}$

and bound charge is conserved.

Since total average charge must be conserved, ie

$$\vec{\nabla}\cdot\langle\vec{J}_0\rangle - \frac{\partial\langle\rho_0\rangle}{\partial t} = 0,$$ and $\langle\vec{J}_0\rangle = \vec{J} + \vec{J}_{bound}$

↑ free current

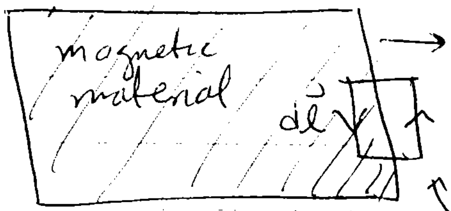
$\langle\rho_0\rangle = \rho + \rho_{bound}$

↑ free charge

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

free charge is also conserved

At a surface of a magnetic material



$\longrightarrow \hat{m}$ outward normal to surface

take $\hat{z} = d\hat{l} \times \hat{m}$ out of page

Amperean loop C bounding surface S of area da

$$c \int_S da\, \hat{z} \cdot (\vec{\nabla} \times \vec{M}) = \int_S da\, \hat{z} \cdot \vec{J}_{bound} = da\, \hat{z} \cdot \vec{J}_{bound}$$

$$= (d\vec{l} \times \hat{m}) \cdot \vec{K}_{bound} \quad \text{as width of loop} \to 0$$

$$= (\hat{m} \times \vec{K}_{bound}) \cdot d\vec{l}$$

But by Stokes theorem

$$c \int_S da\, \hat{z} \cdot (\vec{\nabla} \times \vec{M}) = c \oint_C d\vec{l} \cdot \vec{M} = c\, d\vec{l} \cdot \vec{M} \quad \text{since width} \to 0 \text{ and } \vec{M} = 0 \text{ outside}$$

$$\Rightarrow \quad c\, d\vec{l} \cdot \vec{M} = (\hat{m} \times \vec{K}_{bound}) \cdot d\vec{l} \quad \text{for any } d\vec{l} \text{ in plane of surface}$$

$$\Rightarrow \quad c\vec{M}_t = \hat{m} \times \vec{K}_{bound}$$

where $\vec{M}_t$ is component of $\vec{M}$ tangential to the surface (since $\vec{K}_b$ is in plane of surface, $\vec{m} \times \vec{K}$ is also entirely in the plane of the surface)

$$\Rightarrow \quad c\, \hat{m} \times \vec{M}_t = c\, \hat{m} \times \vec{M} = \hat{m} \times (\hat{m} \times \vec{K}_{bound})$$
$$= -\vec{K}_{bound}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \vec{K}_{bound} &= c\, \vec{M} \times \hat{m} \\ \vec{J}_{bound} &= c\, \vec{\nabla} \times \vec{M} \end{aligned}}$$

Total bound charge vanishes

$$Q_{bound} = \int_V d^3r\ \rho_{bound} + \int_S da\ \sigma_{bound}$$

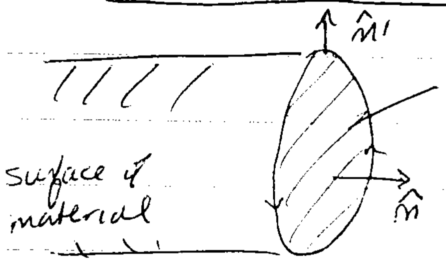
$\uparrow$ vol of dielectric $\quad\leftarrow$ surface of dielectric

$$= \int_V d^3r\ -\vec{\nabla}\cdot\vec{P} + \int da\ \hat{n}\cdot\vec{P}$$

but by Gauss theorem $\displaystyle\int_V d^3r\ \vec{\nabla}\cdot\vec{P} = \int da\ \hat{n}\cdot\vec{P}$

so $\displaystyle Q_{bound} = -\int da\ \hat{n}\cdot\vec{P} + \int da\ \hat{n}\cdot\vec{P} = 0$

Total bound current vanishes



crossectional area of magnetic material

S - bounding curve is C

$\hat{n}$ is normal to crossection
$\hat{n}'$ is normal to surface

total current flowing through S is

$$\int_S da\ \hat{n}\cdot\vec{J}_{bound} + \int_C dl\ \vec{K}_{bound}\cdot\hat{n}$$

$$= c\int_S da\ \hat{n}\cdot(\vec{\nabla}\times\vec{M}) + c\int_C dl\ \hat{n}\cdot(\vec{M}\times\hat{n}')$$

$$= c\int_C d\vec{l}\cdot\vec{M} + c\int_C dl\ \underbrace{(\hat{n}'\times\hat{n})}_{=-\hat{t}}\cdot\vec{M}$$

$= -\hat{t}$ unit tangent, $d\vec{l} = dl\,\hat{t}$

$$= c\int_C d\vec{l}\cdot\vec{M} - c\int_C d\vec{l}\cdot\vec{M} = 0$$

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad\qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \qquad\qquad \vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

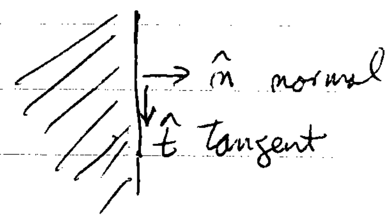
where ρ and $\vec{j}$ are macroscopic charge + current densities
do not include bound charges or currents

$\vec{D} = \vec{E} + 4\pi\vec{P}$, $\vec{P}$ is polarization density

$\vec{H} = \vec{B} - 4\pi\vec{M}$, $\vec{M}$ is magnetization density

Boundary conditions for statics

electrostatics: at surface of a dielectric, or at interface between two different dielectrics



$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \hat{t}\cdot\vec{E}_{above} = \hat{t}\cdot\vec{E}_{below}$

tangential component $\vec{E}$ is continuous

$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \Rightarrow \hat{n}\cdot(\vec{D}_{above} - \vec{D}_{below}) = 4\pi\sigma$

normal component of $\vec{D}$ jumps by $4\pi\sigma$

magnetostatics: at surface or interface of magnetic materials

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \hat{n}\cdot\vec{B}_{above} - \hat{n}\cdot\vec{B}_{below}$

normal component of $\vec{B}$ is continuous

$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} \Rightarrow \hat{t}\cdot(\vec{H}_{above} - \vec{H}_{below}) = \frac{4\pi}{c}\vec{K}$

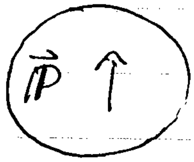
tangential component of $\vec{H}$ jumps by $\frac{4\pi}{c}\vec{K}$

if $\sigma = 0$, ie no free surface charge, then $\hat{n}\cdot\vec{D}$ continuous

if $\vec{K} = 0$, ie no free surface current, then $\hat{t}\cdot\vec{H}$ continuous

Examples

① Uniformly polarized sphere of radius R $\quad \vec{\mathbb{P}} = \mathbb{P}\hat{z}$



bound charge $\quad \rho_b = -\vec{\nabla}\cdot\vec{\mathbb{P}} = 0$ as $\vec{\mathbb{P}}$ constant

$$\sigma_b = \hat{n}\cdot\vec{\mathbb{P}} = \hat{r}\cdot\vec{\mathbb{P}} = \mathbb{P}\cos\theta$$

we saw earlier that a sphere with surface charge $\sigma(\theta) = \sigma_0\cos\theta$ gives an electric field like a pure dipole for $r > R$, and is constant for $r < R$.

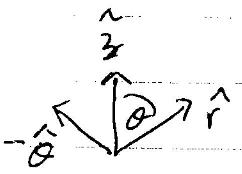
$$\vec{E}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 \mathbb{P}\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r > R \\ -\dfrac{4\pi\mathbb{P}}{3}\,\hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{p} = \frac{4}{3}\pi R^3 \vec{\mathbb{P}}$

check behavior at boundary

Tangential component $\vec{E}$

$$\vec{E}^{\,t}_{above} = \left(\tfrac{4}{3}\pi R^3 \mathbb{P}\right)\frac{\sin\theta\,\hat{\theta}}{R^3} = \tfrac{4}{3}\pi\mathbb{P}\sin\theta\,\hat{\theta}$$



$$\vec{E}^{\,t}_{below} = -\frac{4\pi\mathbb{P}}{3}(\hat{z}\cdot\hat{\theta})\,\hat{\theta} = \tfrac{4}{3}\pi\mathbb{P}\sin\theta\,\hat{\theta}$$

$\Rightarrow$ Tangential component $\vec{E}$ is continuous

normal component of $\vec{D}$

outside: $\vec{\mathbb{P}} = 0 \;\Rightarrow\; \vec{D} = \vec{E}$

$$\Rightarrow \hat{n}\cdot\vec{D} = \hat{r}\cdot\vec{E} = \left(\tfrac{4}{3}\pi R^3 \mathbb{P}\right)\frac{2\cos\theta}{R^3} = \tfrac{8}{3}\pi\,\mathbb{P}\cos\theta$$

inside: $\vec{E} = -\frac{4\pi \vec{P}}{3} \Rightarrow \vec{P} = -\frac{3}{4\pi}\vec{E}$

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \vec{E} - 3\vec{E} = -2\vec{E} = \frac{8\pi}{3} P \hat{z}$$

$$\hat{n} \cdot \vec{D} = \hat{r} \cdot \left(\frac{8\pi}{3} P \hat{z} \right) = \frac{8\pi}{3} P \cos\theta$$

$\Rightarrow$ normal component $\vec{D}$ is continuous

Note: normal component of $\vec{E}$ should jump by $4\pi\sigma_b = 4\pi P\cos\theta$

to check this: $\underset{\text{inside}}{\hat{n} \cdot \vec{E}} = \hat{r} \cdot \left(-\frac{4}{3}\pi P \hat{z} \right) = -\frac{4}{3}\pi P \cos\theta$

$$\hat{n} \cdot \left(\vec{E}^{\text{above}} - \vec{E}^{\text{below}} \right) = \frac{8}{3}\pi P\cos\theta + \frac{4}{3}\pi P\cos\theta$$

$$= \frac{12}{3}\pi P\cos\theta = 4\pi P\cos\theta \overset{4\pi}{=} \sigma_b(\theta)$$

(2) Uniformly magnetized sphere of radius R $\quad \vec{M} = M\hat{z}$

bound current $\quad \vec{J}_b = c\,\vec{\nabla}\times\vec{M} = 0$ as $\vec{M}$ constant

$$\vec{K}_b = c\,\vec{M}\times\hat{n} = cM(\hat{z}\times\hat{r})$$
$$= cM\sin\theta\,\hat{\varphi}$$

we saw earlier that a sphere with surface current $\vec{K}_b = K_0 \sin\theta\,\hat{\theta}$ gives a magnetic field that is pure dipole for $r>R$, and is constant for $r<R$.

$$\vec{B}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 M\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r>R \\ \frac{8}{3}\pi M\,\hat{z} & r<R \end{cases}$$

total dipole moment is $\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$

check behavior at boundary

normal component of $\vec{B}$

$$\hat{n}\cdot\vec{B}^{\,above} = \hat{r}\cdot\vec{B}^{\,above} = \frac{8}{3}\pi M\cos\theta$$

$$\hat{n}\cdot\vec{B}^{\,below} = \hat{r}\cdot\vec{B}^{\,below} = \frac{8}{3}\pi M(\hat{r}\cdot\hat{z}) = \frac{8}{3}\pi M\cos\theta$$

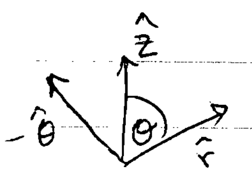
$\Rightarrow$ normal component of $\vec{B}$ is continuous

tangential component of $\vec{H}$

outside: $\vec{M}=0 \Rightarrow \vec{H}=\vec{B}$

$$\vec{H}^{t}_{above} = \left(\frac{4}{3}\pi M\right)\sin\theta\,\hat{\theta}$$

inside: $$\vec{H} = \vec{B} - 4\pi\vec{M} = \vec{B} - 4\pi\left(\frac{3}{8\pi}\vec{B}\right) = \vec{B} - \frac{3}{2}\vec{B} = -\frac{1}{2}\vec{B}$$
$$= -\frac{4\pi}{3}M\hat{z}$$



so $$\vec{H}^{t}_{below} = -\frac{4\pi}{3}M(\hat{z}\cdot\hat{\theta}) = \frac{4\pi}{3}M\sin\theta\,\hat{\theta}$$

$\Rightarrow$ tangential component $\vec{H}$ is continuous

Note: tangential component $\vec{B}$ should jump by $\frac{4\pi}{c}\vec{K}_b = 4\pi M\sin\theta\,\hat{\theta}$

inside:
to check: $$\vec{B}^{t}_{below} = \frac{8}{3}\pi M(\hat{z}\cdot\hat{\theta})\hat{\theta} = -\frac{8}{3}\pi M\sin\theta\,\hat{\theta}$$

$\vec{H}^{t}_{above} = \vec{B}^{t}_{above} \Rightarrow$ $$\vec{B}^{t}_{above} - \vec{B}^{t}_{below} = \frac{4}{3}\pi M\sin\theta\,\hat{\theta} + \frac{8}{3}\pi M\sin\theta\,\hat{\theta}$$

$$= 4\pi M\sin\theta\,\hat{\theta} = \frac{4\pi}{c}\vec{K}_b$$

Linear Materials

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

where ρ and $\vec{j}$ are macroscopic charge + current densities and

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \vec{P} \text{ is polarization density}$$
$$\vec{H} = \vec{B} - 4\pi\vec{M} \qquad \vec{M} \text{ is magnetization density}$$

To close these equations, we will in general need to know how $\vec{P}$ and $\vec{M}$ are related to the $\vec{E}$ and $\vec{B}$ in the material.

In some materials, there can be a finite $\vec{P}$ or $\vec{M}$ even if $\vec{E}$ and $\vec{B}$ are zero:

<u>Ferromagnet</u>: $\vec{M}$ can be non zero even if $\vec{B} = 0$

<u>Ferroelectric</u>: $\vec{P}$ can be non zero even if $\vec{E} = 0$

But more common are <u>linear</u> materials in which, for <u>small</u> $\vec{E}$ and $\vec{B}$, one has $\vec{P} \propto \vec{E}$ and $\vec{M} \propto \vec{B}$.

<u>linear dielectric</u>

$\vec{P} = \chi_e \vec{E}$ χ_e is "electric susceptibility"

$\chi_e > 0$ for statics

$\vec{D} = \vec{E} + 4\pi\vec{P} = (1 + 4\pi\chi_e)\vec{E}$

$\vec{D} = \epsilon\vec{E}$ with $\epsilon = 1 + 4\pi\chi_e$

ϵ is the dielectric constant

<u>linear magnetic materials</u>

$\vec{M} = \chi_m \vec{H}$ χ_m is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$\vec{H} = \vec{B} - 4\pi\vec{M} = \vec{B} - 4\pi\chi_m\vec{H}$

$\vec{B} = (1 + 4\pi\chi_m)\vec{H}$

$\vec{B} = \mu\vec{H}$ with $\mu = 1 + 4\pi\chi_m$

μ is magnetic permeability

For <u>statics</u>, $\chi_e > 0$ and χ_m (or alternatively ϵ and μ) are <u>constants</u> depending on the material.

When we consider <u>dynamics</u> we will see that ϵ becomes a function of <u>frequency</u>.

Clausius - Mossotti equation

Electric susceptibility + atomic polarizability

If a field $\vec{E}_{loc}$ is applied to an atom, it gets polarized

$$\vec{p} = \alpha \vec{E}_{loc}$$

atomic dipole moment — atomic polarizability — "local field" - field the atom sees

α is what one calculates from a microscopic theory

If $\vec{E}_{loc} = \vec{E}$ the average field in the material then electric susceptibility given by

$$\vec{P} = n\vec{p} = n\alpha \vec{E}_{loc} = n\alpha\vec{E} = \chi_e \vec{E}$$

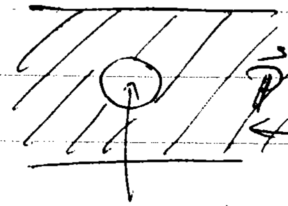
$\Rightarrow \chi_e = n\alpha$ where n = density of atoms

But a more careful consideration shows $\vec{E}_{loc} \neq \vec{E}$

The average field $\vec{E}$ includes the electric field created by the polarized atom itself. $\vec{E}_{loc}$, the local field the atom sees, should exclude its own self field.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom}$$

average field — average field excluding atom — average field of the atom



cut out sphere whose volume is $1/n$ the volume per atom

$\vec{E}_{loc}$ is field excluding the field of the polarized sphere of volume $1/n$.

$\vec{E}_{atom}$ is field of the polarized sphere

$$\vec{E}_{atom} = -\frac{4\pi\vec{\mathbb{P}}}{3} = -\frac{4\pi}{3} n \vec{p}$$

$$\vec{E}_{loc} = \vec{E} - \vec{E}_{atom} = \vec{E} + \frac{4\pi}{3}\vec{\mathbb{P}} = \vec{E} + \frac{4\pi}{3} n \vec{p}$$

$$\vec{p} = \alpha \vec{E}_{loc} = \alpha\left(\vec{E} + \frac{4\pi}{3} n \vec{p}\right) = \alpha\vec{E} + \frac{4\pi}{3} n\alpha\, \vec{p}$$

$$\vec{p} = \frac{\alpha}{1-\frac{4\pi}{3}n\alpha}\vec{E}$$

$$\vec{\mathbb{P}} = n\vec{p} = \frac{\alpha n}{1-\frac{4\pi}{3}n\alpha}\vec{E} = \chi_e \vec{E}$$

$$\boxed{\chi_e = \frac{n\alpha}{1-\frac{4\pi}{3}n\alpha}}$$

or solve for α in terms of ϵ

$$\chi_e = \frac{n\alpha}{1 - \frac{4\pi}{3} n\alpha} \Rightarrow \chi_e - \frac{4\pi}{3} n\alpha\chi_e = \alpha n$$

$$\Rightarrow \alpha = \frac{\chi_e}{n\left(1 + \frac{4\pi}{3}\chi_e\right)}$$

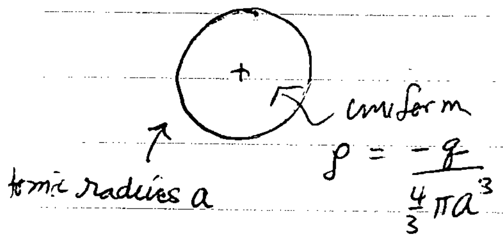
$$\epsilon = 1 + 4\pi\chi_e \Rightarrow \alpha = \frac{\epsilon - 1}{4\pi n} \frac{1}{\left(1 + \frac{\epsilon - 1}{3}\right)}$$

relates atomic polarizability to measured dielectric constant

$$\boxed{\alpha = \frac{3}{4\pi n}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)}$$

Clausius Mossotti or Lorentz-Lorenz equation

simple model for α



field inside is $E(r) = \frac{4\pi\rho}{3} r \hat{r}$

In external field E_0, net forces balance $\Rightarrow qE_0 = q\frac{4\pi\rho d}{3}$

$$p = qd = \frac{3}{4\pi\rho} q E_0 = \frac{3\left(\frac{4}{3}\pi a^3\right)}{4\pi q} q E_0$$

$$= a^3 E_0 \Rightarrow \boxed{\alpha = a^3}$$

$$\chi_e = \frac{na^3}{1 - \frac{4\pi}{3} na^3}$$

if $f = n\frac{4}{3}\pi a^3$ fraction of vol that is occupied by atoms

$$\boxed{\chi_e = \frac{1}{4\pi}\frac{3f}{1-f}}$$