Linear dielectrics

bound charge is proportional to free charge

\[ p_b = - \nabla \cdot \mathbf{P} = - \nabla \cdot (\chi_e \mathbf{E}) = - \nabla \cdot \left( \frac{\chi_e}{\varepsilon} \mathbf{D} \right) \]

if \( \chi_e \) (and hence \( \varepsilon \)) is spatially constant, then

\[ p_b = \frac{-\chi_e}{\varepsilon} \nabla \cdot \mathbf{D} = \frac{-\chi_e}{\varepsilon} 4\pi \rho \]

\[ p_b = -\frac{4\pi \chi_e}{1 + 4\pi \chi_e} \rho \]

when free charge \( \rho = 0 \),

then \( p_b = 0 \)

\[ \text{Total} = p + p_b = p \left( 1 - \frac{4\pi \chi_e}{1 + 4\pi \chi_e} \right) \]

\[ \frac{p}{\varepsilon} = \frac{\text{Total}}{\varepsilon} \]

bound charge "screens" the free charge so the total charge is reduced compared to the free charge.
For linear dielectrics

\[ \nabla \cdot \mathbf{D} = 4\pi \rho \]
\[ \nabla \times \mathbf{E} = 0 \]

\[ \mathbf{D} = \mathbf{E} \varepsilon \Rightarrow \nabla \cdot (\varepsilon \mathbf{E}) = 4\pi \rho \]

If \( \varepsilon \) is constant in space then \( \varepsilon \nabla \cdot \mathbf{E} = 4\pi \rho \)

\[ \nabla \cdot \mathbf{E} = 4\pi \rho / \varepsilon = 4\pi \rho \text{tot} \]

\[ \nabla \times \mathbf{E} = 0 \]

Alternatively, could write \( \mathbf{E} = \mathbf{D} / \varepsilon \)

\[ \Rightarrow \nabla \times (\mathbf{D} / \varepsilon) = 0 \]
\[ \Rightarrow \nabla \times \mathbf{D} = 0 \] when \( \varepsilon \) constant in space

\[ \nabla \cdot \mathbf{D} = 4\pi \rho \]
\[ \nabla \times \mathbf{D} = 0 \]

Complication arises at interface between dielectrics (or between dielectric and vacuum). At interface,
\( \varepsilon \) is not constant \( \Rightarrow \nabla \times \mathbf{D} \neq 0 \).

What we can do is to solve for \( \mathbf{E} \) or \( \mathbf{D} \) inside each dielectric separately, and then use the boundary conditions
\( \mathbf{n} \cdot (\mathbf{D}_{\text{above}} - \mathbf{D}_{\text{below}}) = 4\pi \sigma \)
\( \mathbf{n} \cdot (\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}) = 0 \)

to match solutions across the interfaces.

A similar story holds for linear magnetic materials.
Simple example: parallel plate capacitor filled with a dielectric

\[ \sigma \begin{array}{c} \hline \hline \end{array} \sigma \]  
\[ p \]  
\[ \sigma \text{ free charge} \]

What is \( E \) between plates?

We know \( \vec{E} = \vec{D} = 0 \) outside plates.

Between plates \( \vec{D} = 0 \) as \( \rho = 0 \).

\[ \vec{D} = \vec{D}(x) \hat{x} \Rightarrow \frac{2\vec{D}}{2x} = 0 \Rightarrow \vec{D} \text{ is constant} \]

Boundary conditions:

Left side plate:

\[ \hat{n} = \hat{x}, \quad \vec{D} = 0 \]

\[ x \cdot (\vec{D}_{\text{above}} - \vec{D}_{\text{below}}) = \vec{D} = 4\pi \sigma \]

Right side plate:

\[ \hat{n} = \hat{x}, \quad \vec{D} = 0 \]

\[ x \cdot (\vec{D}_{\text{above}} - \vec{D}_{\text{below}}) = -\vec{D} = 4\pi (-\sigma) \]

\[ \vec{D} = 4\pi \sigma \hat{x} \]

\[ \frac{\vec{E}}{\varepsilon} = \frac{\vec{D}}{\varepsilon} = 4\pi \sigma \hat{x} \]

Electric field reduced by factor \( \frac{1}{\varepsilon} \) as compared to capacitor with vacuum between plates.

See Jackson section 4.9 for more interesting examples
- Dielectric sphere in uniform applied \( E \)

See Jackson section (5.11) for an interesting magnetic b.c. problem
- Spherical permeable shell in uniform applied \( B \)
Point charge within a dielectric sphere

Charge \( q \) at center of dielectric sphere of radius \( R \), dielectric constant \( \varepsilon \)

\[
\nabla \cdot \vec{D} = 4\pi q = \int_S \text{d}a \, \hat{n} \cdot \vec{D} = 4\pi Q \text{ and}
\]

From symmetry: \( \vec{D}(r) = D(r) \hat{r} \)

\[
\int_S \text{d}a \, \hat{n} \cdot \vec{D} = 4\pi r^2 D(r) = 4\pi q
\]

Sphere of radius \( r \)

\[
\vec{D} = \frac{q}{r^2} \hat{r} \quad \text{all } r
\]

\[
\Rightarrow \vec{E}(r) = \begin{cases} \frac{q}{r^2} \hat{r} & r < R \\ \frac{\varepsilon q}{r^2} \hat{r} & r > R \end{cases}
\]

Can check that tangential component of \( \vec{E} \) is continuous and normal component of \( \vec{D} \) is continuous as there is no free \( \sigma \) at surface of dielectric.

Normal component of \( \vec{E} \) jumps by

\[
\hat{n} \cdot (\vec{E}_{\text{above}} - \vec{E}_{\text{below}}) = \frac{q}{R^2} - \frac{\varepsilon q}{R^2} = \frac{q}{R^2} \left( 1 - \frac{1}{\varepsilon} \right) = \frac{q}{R^2} \left( \frac{\varepsilon - 1}{\varepsilon} \right)
\]

\[
= \frac{q}{R^2} \left( \frac{4\pi \kappa_e}{1 + 4\pi \kappa_e} \right) = 4\pi \sigma_{\text{total}} = 4\pi \sigma_b
\]

\[
\Rightarrow \sigma_b = \frac{q}{4\pi R^2} \left( \frac{4\pi \kappa_e}{1 + 4\pi \kappa_e} \right) = \frac{q \kappa_e}{R^2 \varepsilon}
\]

We can check this directly.
\[ \vec{P} = \kappa e \hat{E} = \frac{\kappa e \vec{E}}{r^2} \hat{r} \]

\[ \vec{P} = -\nabla \cdot \vec{P} = -\frac{\kappa e}{\varepsilon} \varepsilon \pi \delta(r) \]

\[ \text{bound charge at origin} \quad q_b = -\frac{\kappa e}{\varepsilon} \pi \]

\[ \text{total charge at origin} \quad q + q_b = \frac{\varepsilon}{\varepsilon} (1 - \frac{4\pi \kappa e}{\varepsilon}) \]

\[ \varepsilon = 1 + 4\pi \kappa e \quad = \frac{\varepsilon}{\varepsilon} (\varepsilon - \frac{4\pi \kappa e}{\varepsilon}) = \frac{\varepsilon}{\varepsilon} \text{ screened charge} \]

\[ \text{at surface} \quad \sigma_b = \hat{\mathbf{n}} \cdot \vec{P} = \frac{\kappa e}{\varepsilon} \frac{\varepsilon}{R^2} \quad \text{agrees with what we get from jump in} \quad \hat{\mathbf{n}} \cdot \vec{E}. \]

\textbf{Note:} \quad \text{inside the dielectric the } \vec{E} \text{ field is that of the screened point charge } \frac{\varepsilon}{\varepsilon} \varepsilon \text{, just that of the free charge } q. \text{ There is no evidence in } \vec{E} \text{ out that the dielectric even exists!}
Now consider same problem but \( q \) is off-center:

\[
\begin{array}{c}
\text{inside: } \vec{V} \cdot \vec{D} = 4\pi \rho \quad \text{where } \rho = q \delta(r - s\hat{z}) \\
\vec{D} = \varepsilon \vec{E} \Rightarrow \vec{V} \cdot \vec{E} = 4\pi \rho / \varepsilon \\
\vec{E} = -\vec{\nabla} \phi \Rightarrow \nabla^2 \phi = -\frac{4\pi \rho}{\varepsilon} = -\frac{4\pi q}{\varepsilon} \delta(r - s\hat{z})
\end{array}
\]

Solution for \( \phi \) will be of the form:

\[
\phi(r) = \frac{q}{\varepsilon|\vec{r} - s\hat{z}|} + F(r)
\]

where 1st term is due to the point charge \( q/\varepsilon \)

and 2nd term satisfies \( \nabla^2 F = 0 \) and will be chosen to get the correct behavior at the boundary of the dielectric.

Since there is cylindrical symmetry about \( \hat{z} \)

we can write:

\[
F(r) = \sum_{l=0}^{\infty} a_l r^l \Phi_l(\cos \Theta)
\]

there are no be terms since \( F \) should not

diverge at the \( r = 0 \) origin.
So inside \( r < R \)

\[
\Phi^i(r) = \frac{q}{r} + \sum_{\ell=0}^{\infty} \alpha_{\ell} r^\ell J_{\ell}(\cos \theta)
\]

From our discussion of electric multipole expansion, we know we can write for \( r > s \),

\[
\frac{1}{|r - s|^2} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left( \frac{s}{r} \right)^\ell J_{\ell}(\cos \theta)
\]

So for \( r > s \) (not true for \( r < s \!\) !)

\[
\Phi^{in}(r) = \sum_{\ell=0}^{\infty} \left( \frac{q}{r} \left( \frac{s}{r} \right)^\ell + \alpha_{\ell} r^\ell \right) J_{\ell}(\cos \theta)
\]

Outside the sphere there is no charge, so \( \nabla \cdot \vec{E} = 0 \) or \( \nabla^2 \Phi = 0 \)

\[
\Rightarrow \Phi^{out}(r) = \sum_{\ell=0}^{\infty} \frac{b_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta)
\]

then there are no \( a_{\ell} \) terms since \( \Phi^{out} \rightarrow 0 \) as \( r \rightarrow \infty \)

To determine the unknown \( a_{\ell} \) and \( b_{\ell} \) we use the boundary conditions at surface of dielectric at \( r = R \).
\( \text{a) Tangential component } \vec{E} \text{ is continuous} \)

\[
\vec{E} = -\frac{\partial \phi}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} = E_r \hat{r} + E_{\theta} \hat{\theta}
\]

\( \Rightarrow \ E_{\theta} \text{ is continuous at } r = R \)

Condition that \( E_{\theta} \) is continuous is the same
condition that \( \phi \) is continuous (check this out for yourself if you are not sure)

\( \Rightarrow \ \phi^{\text{in}}(R, \theta) = \phi^{\text{out}}(R, \theta) \)

\[
\frac{8}{\varepsilon R} \left( \frac{S}{R} \right)^l + a_{2l} R^l = \frac{b \varepsilon}{R^{l+1}}
\]

\( \Rightarrow \ b \varepsilon = \frac{8}{\varepsilon} S^l + a_{2l} R^{2l+1} \)

The normal component \( \vec{D} \) is continuous (since free surface charge \( \sigma = 0 \))

\[
\vec{D} = \varepsilon \vec{E}
\]

\( \Rightarrow \ \varepsilon E_{r}^{\text{in}} = E_{r}^{\text{out}} \)

\[
-\varepsilon \frac{\partial \phi^{\text{in}}}{\partial r} = -\varepsilon \frac{\partial \phi^{\text{out}}}{\partial r}
\]

\( \Rightarrow \ \frac{(l+1) \frac{8}{\varepsilon} \left( \frac{S}{R} \right)^l - l \varepsilon a_{2l} R^{l-1}}{R^2 (\frac{S}{R})^l} = \frac{(l+1) b \varepsilon}{R^{l+2}} \)
\[ \frac{q}{s^e} \frac{e}{e+1} \in a_e R^{2e+1} = b_e \]

Substitute \( b_e \) from previous boundary condition

\[ \frac{q}{s^e} \frac{e}{e+1} e a_e R^{2e+1} = \frac{q}{e} s^e + a_e R^{2e+1} \]

\[ \frac{q}{s^e} \frac{e}{e+1} \left[ 1 - \frac{1}{e} \right] = a_e R^{2e+1} \left[ 1 + \frac{e}{e+1} \frac{e}{e} \right] \]

\[
\begin{align*}
  a_e &= \frac{q}{s^e} \frac{e}{e+1} \frac{\left[ 1 - \frac{1}{e} \right]}{\left[ 1 + \frac{e}{e+1} \right]} \\
  b_e &= \frac{q}{s^e} s^e + a_e R^{2e+1} \\
  &= \frac{q}{s^e} s^e + \frac{q}{s^e} \frac{e}{e+1} \left[ 1 - \frac{1}{e} \right] \\
  &= \frac{q}{s^e} \frac{e}{e+1} \left[ 1 + \frac{e}{e+1} \right] \\
  &= \frac{q}{s^e} \frac{e}{e+1} \left[ \frac{e}{e+1} \right] \\
  &= \frac{q}{s^e} \frac{e}{e+1} \left[ 1 + \frac{e}{e+1} \right]
\end{align*}
\]

\[
\begin{align*}
  b_e &= \frac{q}{s^e} \frac{e}{e+1} \left[ 1 + \frac{e}{e+1} \right] \\
  &= \frac{q}{s^e} \frac{e}{e+1} \left[ \frac{e}{e+1} \right]
\end{align*}
\]
Check the result:

As $s \to 0$, should recover previous answer

For $s = 0$, $a_l = b_l = 0$ for all $l \neq 0$

\[ a_0 = \frac{q}{k} \left[ 1 - \frac{1}{e} \right] \]

\[ b_0 = \frac{q}{r} \]

\[ S_0 \phi^\text{in}(r) = \frac{q}{e r} + \frac{q}{k} \left[ 1 - \frac{1}{e} \right] \]

\[ E^\text{in} = -\nabla \phi^\text{in} = \frac{q}{e r^2} \hat{r} \quad \text{as before} \]

\[ \phi^\text{out}(r) = \frac{q}{r} \]

\[ E^\text{out} = -\nabla \phi^\text{out} = \frac{q}{e r^2} \hat{r} \quad \text{as before} \]

Note: the constant that is the 2nd term in $\phi^\text{in}$

is just what is needed to make $\phi$ continuous at $r = R$. 
let \( \varepsilon \to \infty \) this models a conductor!

again one finds \( a_e = b_e = 0 \) for all \( \ell \neq 0 \)

\[
a_0 = \frac{q}{R}
\]

\[
b_0 = \frac{q}{r}
\]

\[
\phi^\text{in}(r^2) = \frac{q}{\varepsilon r} + \frac{q}{R} \to \frac{q}{R} \quad \text{as} \quad \varepsilon \to \infty
\]

\[
\Rightarrow E^\text{in}(r) = 0 \quad \text{as} \quad \phi^\text{in} \text{ is a constant}
\]

\[
\phi^\text{out}(r) = \frac{q}{r} \Rightarrow E^\text{out} = \frac{q}{r^2}
\]

field outside is like point charge \( q \) at the origin, independent of where \( q \) is inside the sphere.

This is the correct behavior of a conductor.

The mobile charges in the conductor completely screen the \( q \) inside, and leave a uniform

surface charge \( q_b = \frac{q}{4\pi R^2} \) on the surface.
Magnetic states

Bar magnets - $\bar{B} = 0$, $\bar{H}$ fixed and given

\[ \nabla \cdot \bar{B} = 0 \]
\[ \nabla \times \bar{H} = \frac{4\pi}{c} \bar{j} = 0 \]

\[ \nabla \times \bar{H} = 0 \Rightarrow \bar{H} = -\nabla \phi_M \text{ magnetic scalar potential} \]

\[ \bar{B} = \bar{H} + 4\pi \bar{M} \]

\[ \nabla \cdot \bar{B} = \nabla \cdot (\bar{H} + 4\pi \bar{M}) = 0 \]

\[ \nabla \cdot \bar{H} = -\nabla^2 \phi_M = -4\pi \nabla \cdot \bar{M} \]

\[ \nabla^2 \phi_M = 4\pi \nabla \cdot \bar{M} \]

so $\phi_M = -\nabla \cdot \bar{M}$ looks like a magnetic "charge"

$\phi_M$ is source for $\bar{H}$

Also at surfaces of material $\sigma_M = \bar{M} \cdot \bar{n}$ looks like surface charge

\[ \bar{H}(\bar{r}) = \int d^3r' \: \hat{P}_M(\bar{r}') \frac{(\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3} + \oint_{\sigma_M} \nabla^2 \sigma_M(\bar{r}') \frac{(\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3} \]

Field lines for $\bar{H}$ can start and end at sources and sinks given by $\phi_M$ and $\sigma_M$
$\vec{M} = M \hat{z}$

Boundary currents:

$\vec{J}_b = c \vec{\nabla} \times \vec{M} = 0$

$\vec{K}_b = c \vec{M} \times \hat{z}$

$K_{b} = \left\{ \begin{array}{ll} cIM & \text{on side} \\ 0 & \text{on top & bottom} \end{array} \right.$

$\vec{K}_b$ is like a solenoid current.

Field lines of $\vec{B}$ look like:

But $\vec{H}$ is determined as follows:

$\vec{S}_M = -\vec{\nabla} \cdot \vec{M} = 0$

$\vec{S}_M = \vec{H} \cdot \hat{n} = \left\{ \begin{array}{ll} M & \text{on top} \\ -M & \text{on bottom} \end{array} \right.$

Field lines of $\vec{H}$ look like parallel plate capacitor.

Field lines of $\vec{H} = \text{field lines of } \vec{B}$

outside magnet, but they are very different inside the magnet!