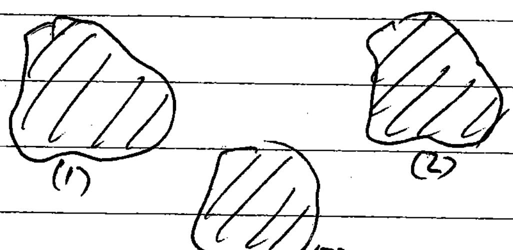
Capacitance

Consider a set of conductors with potential $\phi(\vec{r}) = V_i$ fixed on conductor i



(also need condition on $V(\vec{r}) \to \infty$ if system is not enclosed)

From uniqueness theorem we know that specifying the V_i on each conductor is enough to determine the potential $\phi(\vec{r})$ everywhere. We can write this potential in the following form.

Let $\phi^{(i)}(\vec{r})$ be the solution to the boundary value problem

$$\nabla^2 \phi^{(i)}(r) = 0 \quad \text{and} \quad \phi^{(i)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor } (i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor } (j),\ j \neq i \end{cases}$$

Then by superposition

$$\phi(\vec{r}) = \sum_i V_i \, \phi^{(i)}(\vec{r})$$

is solution to the problem $\nabla^2 \phi = 0$ and $\phi(\vec{r}) = V_i$ for $\vec{r}$ on surface of conductor (i)

The surface charge density at $\vec{r}$ on surface of conductor (i) is

$$\sigma^{(i)}(\vec{r}) = \frac{-1}{4\pi} \frac{\partial \phi(\vec{r})}{\partial n} = -\frac{1}{4\pi} \sum_j V_j \frac{\partial \phi^{(j)}(\vec{r})}{\partial n}$$

where $\dfrac{\partial \phi}{\partial n} = (\vec{\nabla} \phi) \cdot \hat{n}$ is the derivative normal to the surface at point $\vec{r}$.

The total charge on conductor (i) is

$$Q_i = \int_{S_i} da\, \sigma^{(i)}(\vec{r}) = -\frac{1}{4\pi}\sum_j V_j \int_{S_i} da\, \frac{\partial \Phi^{(j)}}{\partial n}$$

surface of conductor(i)

Define $C_{ij} \equiv -\frac{1}{4\pi}\int_{S_i} da\, \frac{\partial \phi^{(j)}}{\partial n}$ the C_{ij} depend only on the <u>geometry</u> of the conductors

Then we have

$$\boxed{Q_i = \sum_j C_{ij} V_j}$$

<u>C_{ij} is the capacitance matrix</u>

The charge on conductor (i) is a linear function of the potentials V_j on the conductors (j)

Since we know that specifying the Q_i that is on each conductor will uniquely determine $\phi(\vec{r})$ and hence the potential V_i on each conductor, the capacitance matrix is invertable

$$V_i = \sum_j [C^{-1}]_{ij} Q_j$$

The electrostatic energy of the conductors is then

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\, \Phi = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_{i,j} C_{ij} V_i V_j$$

Common to define capacitance of two conductors by

$$C = \frac{Q}{V_1 - V_2}$$ when conductor (1) has charge Q
conductor (2) has charge $-Q$
$V_1 - V_2$ is potential difference between the two conductors.
all other conductors fixed at $V_i = 0$

We can determine C in terms of the elements of the matrix C_{ij}

$$\left.\begin{aligned} Q &= C_{11}V_1 + C_{12}V_2 \\ -Q &= C_{21}V_1 + C_{22}V_2 \end{aligned}\right\} \Rightarrow V_2 = -\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)V_1$$

$$\Rightarrow Q = \left[C_{11} - C_{12}\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)\right]V_1$$

$$V_1 - V_2 = \left[1 + \left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)\right]V_1$$

$$C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12}\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)}{1 + \left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)}$$

$$\boxed{C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}}$$

Capacitance can also be defined when the space between the conductors is filled with a dielectric ϵ. In this case, if Q_i is the free charge, then Q_i/ϵ is the effective total charge to use in computing ϕ.

$$\Rightarrow \frac{Q_i}{\epsilon} = \sum_j C_{ij}^{(0)} V_j$$

where $C_{ij}^{(0)}$ are capacitances appropriate to a vacuum between the conductors

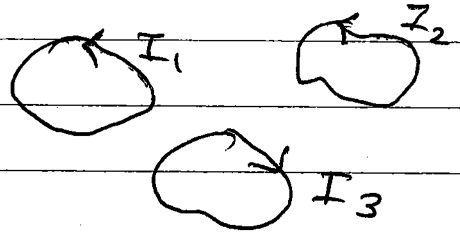
$$\Rightarrow Q_i = \sum_j \epsilon\, C_{ij}^{(0)} V_j$$

$$= \sum_j C_{ij} V_j \quad \text{where } C_{ij} = \epsilon\, C_{ij}^{(0)}$$

the capacitance is increased by a factor the dielectric constant ϵ.

Inductance

Consider a set of current carrying loops C_i with currents I_i



In Coulomb gauge, we can write the magnetic vector potential $\vec{A}$ from these current loops as

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \sum_i \frac{I_i}{c} \oint_{C_i} \frac{d\vec{l}\,'}{|\vec{r}-\vec{r}\,'|}$$

↑ integrate over loop C_i
integration variable is $\vec{r}\,'$

The magnetic flux through loop i is

$$\Phi_i = \int_{S_i} da \, \hat{n}\cdot\vec{B} = \int_{S_i} da \, \hat{n}\cdot\vec{\nabla}\times\vec{A} = \oint_{C_i} d\vec{l}\cdot\vec{A}$$

↑ surface bounded by loop C_i

$$\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i}\oint_{C_j} \frac{d\vec{l}\cdot d\vec{l}\,'}{|\vec{r}-\vec{r}\,'|}$$

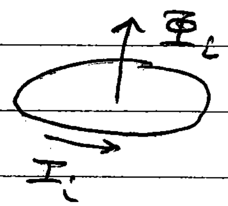
$$\boxed{\Phi_i \equiv c\sum_j M_{ij} I_j}$$

where $M_{ij} = \oint_i \oint_j \frac{d\vec{l}_i \cdot d\vec{l}_j}{c^2|\vec{r}-\vec{r}\,'|}$ ← pure geometrical quantity

is the mutual inductance of loops (i) and (j). $M_{ij} = M_{ji}$

$L_i \equiv M_{ii}$ is <u>self-inductance</u> of loop (i)

The sign convention in the above is that, Φ_i is computed in direction given by right hand rule, according to the direction taken for current in loop (i)



Magnetostatic energy

$$\mathcal{E} = \frac{1}{2c}\int d^3r\, \vec{j}\cdot\vec{A} = \frac{1}{2c}\sum_i \oint_{C_i} d\vec{l}\cdot\vec{A}\; I_i$$

$$= \frac{1}{2c}\sum_i \Phi_i I_i$$

$$\mathcal{E} = \frac{1}{2}\sum_{i,j} M_{ij} I_i I_j$$

Force and torque on electric dipoles

...ed charge distribution $\rho(\vec{r})$ with net charge $\int d^3r\, \rho = 0$

force on ρ in slowly varying electric field $\vec{E}$ is

$$\vec{F} = \int d^3r\, \rho(\vec{r})\, \vec{E}(\vec{r})$$

define $\vec{r} = \vec{r}_0 + \vec{r}\,'$ where $\vec{r}_0$ is some fixed reference point in center of charge distr ρ, and $\vec{r}\,'$ is distance relative to $\vec{r}_0$

$$\vec{F} = \int d^3r'\, \rho(\vec{r}\,')\, \vec{E}(\vec{r}_0 + \vec{r}\,')$$

since $\vec{E}$ is slowly varying on length scale where $\rho \neq 0$, we expand

$$\vec{F} \cong \int d^3r'\, \rho(\vec{r}\,') \left[\vec{E}(\vec{r}_0) + (\vec{r}\,' \cdot \vec{\nabla}) \vec{E}(\vec{r}_0) \right] + \cdots$$

$$= \vec{E}(\vec{r}_0) \int d^3r'\, \rho(\vec{r}\,') + \left(\int d^3r'\, \rho(r')\, \vec{r}\,' \cdot \vec{\nabla} \right) \vec{E}(\vec{r}_0)$$

$$= 0 + (\vec{p} \cdot \vec{\nabla}) \vec{E}(\vec{r}_0)$$

$$\boxed{\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} = \sum_{\alpha=1}^{3} p_\alpha \frac{\partial \vec{E}}{\partial r_\alpha}}$$

For $\vec{E}$ = constant, $\vec{F} = 0$

Torque on ρ is ~~[illegible]~~

$$\vec{N} = \int d^3r\, \rho(\vec{r})\, \vec{r} \times \vec{E}(\vec{r}) \cong \int d^3r\, \rho(\vec{r})\, \vec{r} \times \left[\vec{E}(\vec{r}_0) + \cdots\right]$$

to lowest order $\boxed{\vec{N} = \vec{p} \times \vec{E}}$

Force and torque on magnetic dipoles

localized magnetostatic current distribution $\vec{j}(\vec{r})$

$$\vec{F} = \frac{1}{c}\int d^3r\, \vec{j} \times \vec{B}$$

expand about center of current $\vec{r}_0$

$$\vec{B}(\vec{r}) \simeq \vec{B}(\vec{r}_0) + (\vec{r}' \cdot \vec{\nabla})\vec{B}(\vec{r}_0) + \cdots$$

$$\vec{F} = \frac{1}{c}\left[\int d^3r'\, \vec{j}(\vec{r}')\right] \times \vec{B}(\vec{r}_0) + \frac{1}{c}\int d^3r'\, \vec{j}(\vec{r}') \times (\vec{r}' \cdot \vec{\nabla})\vec{B}(\vec{r}_0)$$

from discussion of magnetic dipole approx we had $\int d^3r\, \vec{j} = 0$ for magnetostatics where $\vec{\nabla} \cdot \vec{j} = 0$. So 1st term vanishes. The 2nd term can be written as

$$F_\alpha = \frac{\epsilon_{\alpha\beta\gamma}}{c}\int d^3r'\, j_\beta\, r'_\delta\, \partial_\delta B_\gamma$$

for magnetostatics; see magnetic dipole derivation

we need the tensor $\frac{1}{c}\int d^3r'\, j_\beta\, r'_\delta = -\frac{1}{c}\int d^3r'\, r'_\beta\, j_\delta$

$$= \frac{1}{2c}\int d^3r' \left[j_\beta r'_\delta - r'_\beta j_\delta\right]$$

$$= -m_\sigma\, \epsilon_{\sigma\beta\delta}$$

magnetic dipole $\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r} \times \vec{j}$

$$F_\alpha = \varepsilon_{\alpha\beta\gamma}\,\varepsilon_{\sigma\beta\delta}(-m_\sigma)\,\partial_\delta B_\gamma$$

$$= -\left(\delta_{\alpha\sigma}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\sigma\gamma}\right) m_\sigma \partial_\delta B_\gamma$$

$$= \vec{\nabla}_\alpha(\vec{m}\cdot\vec{B}) - \vec{m}_\alpha \vec{\nabla}\cdot\vec{B}$$

$$\boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})} \qquad \text{as } \vec{\nabla}\cdot\vec{B} = 0$$

torque on $\vec{j}$ is

$$\vec{N} = \frac{1}{c}\int d^3r\, \vec{r}\times(\vec{j}\times\vec{B})$$ to lowest order, $\vec{B} = \vec{B}(\vec{r}_0)$ is const over region where $\vec{j}\neq 0$

$$= \frac{1}{c}\int d^3r\left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{B}\,(\vec{r}\cdot\vec{j})\right]$$

2nd term $=0$ as follows

$$\int d^3r\; \vec{r}\cdot\vec{j} = \int d^3r\; \vec{j}\cdot\vec{\nabla}\left(\frac{r^2}{2}\right) \qquad \text{as } \vec{\nabla}\left(\frac{r^2}{2}\right) = \vec{r}$$

$$= -\int d^3r\,(\vec{\nabla}\cdot\vec{j})\left(\frac{r^2}{2}\right)$$ integrate by parts. Surface term $\to 0$ as $\vec{j}$ is localized

$$= 0 \qquad \text{as } \vec{\nabla}\cdot\vec{j} = 0 \text{ in magnetostatics}$$

1st term involves

see derivation of magnetic dipole approx

$$\int d^3r\; \vec{j}\,\vec{r} = -\int d^3r\; \vec{r}\,\vec{j} = \frac{1}{2}\int d^3r\left[\vec{j}\,\vec{r} - \vec{r}\,\vec{j}\right]$$

So

$$\vec{N} = \frac{1}{2c}\int d^3r\left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{r}\,(\vec{j}\cdot\vec{B})\right]$$

$$\vec{N} = \frac{1}{2c}\int d^3r \left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{r}\,(\vec{j}\cdot\vec{B}) \right]$$

$$= \cancel{\vec{m}\times\vec{B}}$$

$$= \frac{1}{2c}\int d^3r \,(\vec{r}\times\vec{j}\,)\times\vec{B}$$

$$\boxed{\vec{N} = \vec{m}\times\vec{B}}$$

Electrostatic energy of interaction

$\mathcal{E} = \frac{1}{8\pi} \int d^3r \; E^2$

Suppose the charge density ρ that produces $\vec{E}$ can be broken into two pieces, $\rho = \rho_1 + \rho_2$ with $\vec{E} = \vec{E}_1 + \vec{E}_2$ where $\vec{\nabla}\cdot\vec{E}_1 = 4\pi\rho_1$ and $\vec{\nabla}\cdot\vec{E}_2 = 4\pi\rho_2$

Then

$$\mathcal{E} = \frac{1}{8\pi} \int d^3r \left[E_1^2 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2 \right]$$

"self-energy" of ρ_1 "self energy" of ρ_2 "interaction" energy of ρ_1 with ρ_2

$$\mathcal{E}_{int} = \frac{1}{4\pi} \int d^3r \; \vec{E}_1\cdot\vec{E}_2$$

$$= \int d^3r \; \rho_1 \Phi_2 = \int d^3r \; \rho_2 \Phi_1$$

where $\vec{E}_1 = -\vec{\nabla}\Phi_1$, $\vec{E}_2 = -\vec{\nabla}\Phi_2$, by similar manipulations as earlier

integrals are over <u>all</u> space

Apply to the interaction energy of a dipole in an external $\vec{E}$ field

$$\mathcal{E}_{int} = \int d^3r \; \rho_1 \Phi_2$$

Φ_2 ← potential of external $\vec{E}$ field

ρ_1 ← charge distribution of dipole

Assuming ϕ_2 varys slowly on length scale of ρ_1 then we can expand $\phi_2(\vec{r}) = \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0)\cdot\vec{\nabla}\phi_2(\vec{r}_0)$

where $\vec{r}_0$ is the center of mass or any other convenient reference positeon within ρ_1

$$\mathcal{E}_{int} = \int d^3r\, \rho_1(\vec{r})\left[\phi_2(\vec{r}_0) + (\vec{r}-\vec{r}_0)\cdot\vec{\nabla}\phi_2(\vec{r}_0)\right]$$

$$= q\,\phi_2(\vec{r}_0) + \left[\int d^3r\, \rho_1(\vec{r})(\vec{r}-\vec{r}_0)\right]\cdot\vec{\nabla}\phi_2(\vec{r}_0)$$

$$= q\,\phi_2(\vec{r}_0) - \vec{p}\cdot\vec{E}$$

where q is total charge in ρ_1 and $\vec{p}$ is dipole moment with respect to $\vec{r}_0$. $\vec{E} = -\vec{\nabla}\phi_2$ is external $\vec{E}$-field

For a neutral charge distribution $q=0$, and $\vec{p}$ is independent of the origin about which it is computed, so

$$\boxed{\mathcal{E}_{int} = -\vec{p}\cdot\vec{E}}$$

← does not include the energy needed to make the dipole or to make $\vec{E}$.

$\mathcal{E}_{int}$ is lowest when $\vec{p} \parallel \vec{E}$

⇒ in thermal ensemble, dipoles tend to allign parallel to an applied $\vec{E}$.

Energy of magnetic dipole in external field

We had that the force on the dipole was

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

If we regard the force as coming from the gradient of a potential energy U then $\vec{F} = -\vec{\nabla}U \Rightarrow$

$$\boxed{U = -\vec{m}\cdot\vec{B}}$$

or equivalently, energy = work done to move dipole into position from ∞

$$W = -\int_{\infty} \vec{F}\cdot d\vec{l} = -\int_{\infty}^{\vec{r}} \vec{\nabla}(m\cdot\vec{B})\cdot d\vec{l} = -\vec{m}\cdot\vec{B}(\vec{r})$$

This is the correct energy to use in cases where $\vec{m}$ is due to intrinsic magnetic moments of atom or molecule - say from electron or nuclear spin. For a thermal ensemble magnetic moments tend to allign $\parallel$ to $\vec{B}$.

The answer comes out quite differently if we are talking about a magnetic moment produced by a classical current loop. To see this, consider what we would get if we tried to do the calculation in a similar way to how we did it the the energy of an electric dipole in an electric field...

Magnetostatic energy of interaction

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\ B^2$$

Suppose current $\vec{J}$ that produces $\vec{B}$ can be divided $\vec{J} = \vec{J}_1 + \vec{J}_2$ with $\vec{B} = \vec{B}_1 + \vec{B}_2$ where $\vec{\nabla}\times\vec{B}_1 = \frac{4\pi}{c}\vec{J}_1$ and $\vec{\nabla}\times\vec{B}_2 = \frac{4\pi}{c}\vec{J}_2$. Then

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\ \Big[\underbrace{B_1^2}_{\text{self energy of } \vec{J}_1} + \underbrace{B_2^2}_{\text{self energy of } \vec{J}_2} + \underbrace{2\vec{B}_1\cdot\vec{B}_2}_{\text{interaction energy of } \vec{J}_1 \text{ with } \vec{J}_2}\Big]$$

$$\mathcal{E}_{int} = \frac{1}{4\pi}\int d^3r\ \vec{B}_1\cdot\vec{B}_2$$

$$= \frac{1}{c}\int d^3r\ \vec{J}_1\cdot\vec{A}_2 = \frac{1}{c}\int d^3r\ \vec{J}_2\cdot\vec{A}_1$$

where $\vec{B}_1 = \vec{\nabla}\times\vec{A}_1$, $\vec{B}_2 = \vec{\nabla}\times\vec{A}_2$, by similar manipulations as earlier

integrals are over all space

Apply to the interaction energy of a magnetic dipole in an external $\vec{B}$ field.

$$\mathcal{E}_{int} = \frac{1}{c}\int d^3r\ \underbrace{\vec{J}_1}_{\text{current distribution of dipole}}\cdot\underbrace{\vec{A}_2}_{\text{vector potential of external } \vec{B} \text{ field}}$$

Assuming $\vec{A}$ varies slowly on length scale of $\vec{j}$, then expand $A_i(\vec{r}) = A_i(\vec{r}_0) + (\vec{r}-\vec{r}_0)\cdot\vec{\nabla}A_i(\vec{r}_0)$

$$\mathcal{E}_{int} = \frac{1}{c}\int d^3r\ \vec{j}_1\cdot\vec{A}(\vec{r}_0)$$

$$+\frac{1}{c}\int d^3r\ \sum_i j_{1i}(r-r_0)_j\,\partial_j A_i(\vec{r}_0)$$

~~shift origin so origin at $\vec{r}_0$ ~ $\vec{r}$ now measures distance~~

From ~~magnetostatic~~ computation of magnetic dipole moment we had $\int d^3r\ \vec{j} = 0$ for magnetostatics

$\Rightarrow$ 1st term above vanishes. So does the piece of 2nd term $\left(\int d^3r\ j_{1i}\right) r_{0j}\,\partial_j A_i(\vec{r}_0)$

We are left with

$$\mathcal{E}_{int} = \left[\frac{1}{c}\int d^3r\ j_{1i}\,r_j\right]\partial_j A_i(\vec{r}_0)$$

summation over repeated indices is implied

From computation of magnetic dipole approx we had

$$\int d^3r\ j_{1i}r_j = -\int d^3r\ j_{1j}r_i$$

$$= \frac{1}{2}\int d^3r\left[j_{1i}r_j - j_{1j}r_i\right]$$

Recall:
$\vec{m} = \frac{1}{2c}\int d^3r\ \vec{r}\times\vec{j}$

$$= \frac{1}{2}\,\varepsilon_{kij}\int d^3r\ (\vec{j}\times\vec{r})_k$$

$$\Rightarrow \frac{1}{c}\int d^3r\ j_{1j}r_i = -\varepsilon_{kij}\,m_k$$ ← mag dipole moment

$$\mathcal{E}_{int} = -m_k \varepsilon_{kij} \partial_j A_i = m_k \varepsilon_{kji} \partial_j A_i$$

$$= \vec{m} \cdot (\vec{\nabla} \times \vec{A}) = \vec{m} \cdot \vec{B} = \mathcal{E}_{int}$$

This is <u>opposite in sign</u> to what we found earlier!

<u>Why the difference</u>?

① When we integrate the work done against the magnetostatic force to move $\vec{m}$ into position from infinity we found the energy

$$U = -\vec{m} \cdot \vec{B}$$

② When we compute the interaction energy from

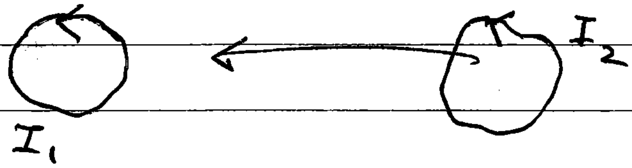
$$\mathcal{E}_{int} = \frac{1}{c} \int d^3r\, \vec{j}_1 \cdot \vec{A}_2 = \frac{1}{c^2} \int d^3r \int d^3r' \frac{\vec{j}_1(\vec{r}) \cdot \vec{j}_2(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

we found the energy $\mathcal{E}_{int} = +\vec{m} \cdot \vec{B}$

To see which is correct, let us consider computing the interaction energy ② directly via method ①.

Consider two loops with currents I_1 and I_2

What is the work done to move loop 2 in from infinity to its final position with respect to loop 1?



Magnetostatic force on loop 2 due to loop 1 is

$$\vec{F} = \frac{I_2}{c} \oint_2 d\vec{l}_2 \times \vec{B}_1$$ Lorentz force

$\vec{B}_1$ is magnetic field from loop 1

$$\vec{B}_1(\vec{r}) = \frac{I_1}{c} \oint_1 d\vec{l}_1 \times \frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3}$$ Biot-Savart law

$$F = \frac{I_1 I_2}{c^2} \oint_2 \oint_1 d\vec{l}_2 \times \frac{(d\vec{l}_1 \times (\vec{r}_2-\vec{r}_1))}{|\vec{r}_2-\vec{r}_1|^3}$$

use triple product rule

$$d\vec{l}_2 \times [d\vec{l}_1 \times (\vec{r}_2-\vec{r}_1)] = d\vec{l}_1 [d\vec{l}_2 \cdot (\vec{r}_2-\vec{r}_1)] - (\vec{r}_2-\vec{r}_1)(d\vec{l}_1 \cdot d\vec{l}_2)$$

from the 1st term

$$\oint_2 d\vec{l}_2 \cdot \frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3} = -\oint_2 d\vec{l}_2 \cdot \vec{\nabla}_2 \left(\frac{1}{|\vec{r}_2-\vec{r}_1|}\right) = 0$$

as integral of gradient around closed loop always vanishes!

So

$$\vec{F} = -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

write $\vec{r}_2 = \vec{R} + \delta\vec{r}_2$ where $\vec{R}$ is center of loop 2

use $$\frac{\vec{R} + \delta\vec{r}_2 - \vec{r}_1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|^3} = -\vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|^3} \right)$$

$$\vec{F} = \frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|^3} \right)$$

to move loop 2 we need to apply a $\overset{\text{mechanical}}{\text{force}}$ equal and opposite to the above magnetostatic force. Therefore the work we do in moving loop 2 from infinity to its final position at $\vec{R}_0$ is

$$W_{mech} = -\int_{\infty}^{\vec{R}_0} \vec{F} \cdot d\vec{R} = -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \int_{\infty}^{\vec{R}_0} d\vec{R} \cdot \vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$$

$$= -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|}$$ where $\vec{r}_2 = \vec{R}_0 + \delta\vec{r}_2$

$$= -\frac{1}{c^2} \int d^3r_1 \int d^3r_2 \frac{\vec{j}_1(\vec{r}_1) \cdot \vec{j}_2(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|}$$

note the minus sign!

$$= -M_{12} I_1 I_2$$

mutual inductance

this is just the negative of the interaction energy!!

Why the minus sign!

The minus sign we have here is the same minus sign we got when we found $U = -\vec{m}\cdot\vec{B}$ by integrating the force on the magnetic dipole.

Why dont we get $+\frac{1}{c^2}\int d^3r_1\, d^3r_2\, \frac{\vec{J}_1(r_1)\cdot\vec{J}_2(r_2)}{|\vec{r}_2-\vec{r}_1|}$

with the plus sign we expect from $\mathcal{E} = \frac{1}{8\pi}\int d^3r\, B^2$?

Answer: we have left something out!

Faraday's Law - when we move loop 2, the magnetic flux through loop 2 changes. This $\frac{d\Phi}{dt}$ creates an emf $= \oint d\vec{l}\cdot\vec{E}$ around the loop that would tend to change the current in the loop. If we are to keep the current fixed at constant I_2 then there must be a battery in the loop that does work to counter this induced emf ("electromotive force"). Similarly, the flux through loop 1 is changing and a battery does work to keep I_1 constant. We need to add this work done by the batteries to the mechanical work computed above.

emf induced in loop 1 $\quad \mathcal{E}_1 = \oint_1 d\vec{l}_1\cdot\vec{E}_2$

emf induced in loop 2 $\quad \mathcal{E}_2 = \oint_2 d\vec{l}_1\cdot\vec{E}_1$

integrations in direction of current

Faraday $\quad \mathcal{E}_1 = \frac{-d\Phi_1}{c\,dt} \qquad \Phi_1 = \text{flux through loop 1}$

$\mathcal{E}_2 = \frac{-d\Phi_2}{c\,dt} \qquad \Phi_2 = \text{flux through loop 2}$

To keep the current constant, the batteries need to provide an emf that counters these Faraday induced emf's. The work done by the batteries per unit time is therefore

$$\frac{dW_{battery}}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2$$

(check units: $\mathcal{E} I$ is [length]·[E]·[q/s]
= [length]·[force/s]
= energy/s)

$$\frac{dW_{battery}}{dt} = \frac{d\Phi_1}{c\,dt} I_1 + \frac{d\Phi_2}{c\,dt} I_2$$

$$W_{battery} = \int_0^T dt \left(\frac{d\Phi_1}{c\,dt} I_1 + \frac{d\Phi_2}{c\,dt} I_2 \right)$$

where $t=0$ loop 2 is at infinity
$t=T$ loop 2 is at final position
I_1, I_2 kept constant as loop moves

$$W_{battery} = \frac{1}{c}\Phi_1 I_1 + \frac{1}{c}\Phi_2 I_2$$

where Φ_1 and Φ_2 are fluxes in final position, and we assumed that fluxes $= 0$ at infinity

$$\Phi_1 = c M_{12} I_2$$

$$\Phi_2 = c M_{21} I = c M_{12} I_1 \quad \text{as } M_{12} = M_{21}$$

$$\Rightarrow W_{battery} = 2 M_{12} I_1 I_2$$

add this to the mechanical work

$$W_{total} = W_{mech} + W_{battery} = -M_{12} I_1 I_2 + 2 M_{12} I_1 I_2$$

$$= M_{12} I_1 I_2 = + \frac{1}{c^2} \int d^3r_1 \, d^3r_2 \, \frac{\vec{j}_1(\vec{r}_1) \cdot \vec{j}_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$$

we get back the correct interaction energy!

<u>Conclusion</u>: The magnetostatic interaction energy $\frac{1}{c^2} \int d^3r_1 \, d^3r_2 \, \frac{\vec{j}_1(\vec{r}_1) \cdot \vec{j}_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$

includes the work done to maintain the currents stationary as the current distributions move.

When we computed the interaction energy of a <u>current loop</u> dipole $\vec{m}$ and find

$$\mathcal{E}_{int} = + \vec{m} \cdot \vec{B}$$

this includes the energy needed to maintain the constant current producing the constant $\vec{m}$

When we integrated the force on the dipole to find the potential energy

$$U = - \vec{m} \cdot \vec{B}$$

this <u>did not</u> include the energy needed to maintain the constant current that creates $\vec{m}$.

This is the correct energy expression to use when $\vec{m}$ comes from <u>intrinsic</u> <u>magnetic</u> <u>moments</u> due to particles intrinsic spin, which cannot be viewed as arising from a current loop!