Capacitance

Consider a set of conductors with potential \( \phi(\vec{r}) = V_i \) fixed on conductor \( i \):

(also need condition on \( V(\vec{r}) = \infty \) if system is not enclosed)

From uniqueness theorem we know that specifying the \( V_i \) on each conductor is enough to determine the potential \( \phi(\vec{r}) \) everywhere. We can write this potential in the following form:

Let \( \phi^{(c)}(\vec{r}) \) be the solution to the boundary value problem

\[ \nabla^2 \phi^{(c)}(\vec{r}) = 0 \quad \text{and} \quad \phi^{(c)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor } (i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor } (j), \ j \neq i \end{cases} \]

Then by superposition

\[ \phi(\vec{r}) = \sum_i V_i \phi^{(c)}(\vec{r}) \]

is solution to the problem \( \nabla^2 \phi = 0 \) and \( \phi(\vec{r}) = V_i \) for \( \vec{r} \) on surface of conductor \( (i) \)

The surface charge density at \( \vec{r} \) on surface of conductor \( (i) \) is

\[ \sigma^{(c)}(\vec{r}) = \frac{1}{4\pi} \frac{\partial \phi(\vec{r})}{\partial n} = -\frac{1}{4\pi} \sum_j V_j \frac{\partial \phi^{(c)}(\vec{r})}{\partial n} \]

where \( \frac{\partial \phi}{\partial n} = (\nabla \phi) \cdot \hat{m} \) is the derivative normal to the surface at point \( \vec{r} \).
The total charge on conductor ($i$) is

$$Q_i = \int_{S_i} d\sigma \sigma^{(i)}(\vec{r}) = -\frac{1}{4\pi} \sum_j V_j \int_{S_i} d\sigma \frac{\partial \phi^{(i)}}{\partial m}$$

$\uparrow$

Surface of conductor ($i$)

Define $C_{ij} = -\frac{1}{4\pi} \int_{S_i} d\sigma \frac{\partial \phi^{(i)}}{\partial m}$

the $C_{ij}$ depend only on the geometry of the conductors

Then we have

$$Q_i = \sum_j C_{ij} V_j$$

$C_{ij}$ is the capacitance matrix

The charge on conductor ($i$) is a linear function of the potentials $V_j$ on the conductors ($j$)

Since we know that specifying the $Q_i$ that is on each conductor will uniquely determine $\phi(\vec{r})$ and hence the potential $V_i$ on each conductor, the capacitance matrix is invertible

$$V_i = \sum_j [C^{-1}]_{ij} Q_j$$

The electrostatic energy of the conductors is then

$$E = \frac{1}{2} \int d^3r \rho \phi = \frac{1}{2} \sum_i Q_i V_i = \frac{1}{2} \sum_{i,j} C_{ij} V_i V_j$$
Common to define capacitance of two conductors by
\[ C = \frac{Q}{V_1 - V_2} \]
when conductor (1) has charge \( Q \)
and conductor (2) has charge \(-Q\)
\( V_1 - V_2 \) is potential difference
between the two conductors.
all other conductors fixed at \( V_c = 0 \)
We can determine \( C \) in terms of the elements of the
matrix \( C_{ij} \):
\[ Q = C_{11}V_1 + C_{12}V_2 \]
\[ -Q = C_{21}V_1 + C_{22}V_2 \]
\[ \Rightarrow \begin{align*}
Q &= C_{11}V_1 + C_{12}V_2 \\
-Q &= C_{21}V_1 + C_{22}V_2
\end{align*} \]
\[ \Rightarrow \begin{align*}
Q &= \left[ C_{11} - C_{12} \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1 \\
V_1 - V_2 &= \left[ 1 + \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1
\end{align*} \]
\[ C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12} \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)}{1 + \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)} \]
\[ C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}} \]

Capacitance can also be defined when the space between the conductors is filled with a dielectric \( \varepsilon \).
In this case, if \( Q_i \) is the free charge, then \( Q = Q_i/\varepsilon \) is
the effective total charge to use in computing \( \Phi \).
\[ \Phi_i = \frac{1}{\varepsilon} \sum_j C_{ij} V_j \]

where \( C_{ij}^{(0)} \) are capacitances appropriate to a vacuum between the conductors.

\[ A_i = \sum_j \varepsilon C_{ij}^{(0)} V_j \]

\[ = \sum_j C_{ij} V_j \]

where \( C_{ij} = \varepsilon C_{ij}^{(0)} \)

the capacitance is increased by a factor the dielectric constant \( \varepsilon \).
Consider a set of current carrying loops \( C_i \) with currents \( I_i \).

In Coulomb gauge, we can write the magnetic vector potential \( \mathbf{A} \) from these current loops as

\[
\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int \! d^3 r' \, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \sum_i \frac{I_i}{c} \oint_{C_i} \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|}
\]

Integrate over loop \( C_i \).

The magnetic flux through loop \( C_i \) is

\[
\Phi_i = \oint_{S_i} \mathbf{n} \cdot \mathbf{B} = \oint_{S_i} \mathbf{n} \cdot \nabla \times \mathbf{A} = \oint_{C_i} \mathbf{dl} \cdot \mathbf{A}
\]

The surface bounded by loop \( C_i \) is a pure geometrical quantity.

\[
\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i \cap C_j} \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|}
\]

where \( M_{ij} = \oint_{\mathbf{C}_i \cap \mathbf{C}_j} \frac{d\mathbf{l}_i \cdot d\mathbf{l}_j}{|\mathbf{r} - \mathbf{r}'|} \)

is the mutual inductance of loops \( C_i \) and \( C_j \). \( M_{ij} = M_{ji} \)
$L_i = M_{ii} \text{ is self-inductance of loop (i)}$

The sign convention in the above is that

$\Phi_i$ is computed in direction given by right hand rule, according to the direction taken for current in loop (i)

\[ \Phi_i \]

---

**Magnetic energy**

\[
E = \frac{1}{2} \oint \mathbf{d} s \cdot \mathbf{A} = \frac{1}{2e} \sum_i \oint \mathbf{d} l \cdot \mathbf{A}_i \text{ } I_i
\]

\[
= \frac{1}{2e} \sum_i \Phi_i \text{ } I_i
\]

\[
E = \frac{1}{2} \sum_{i,j} M_{ij} \text{ } I_i \text{ } I_j
\]
force and torque on electric dipoles

net charge distribution \( q(r) \) with net charge \( \int d^3r' q = 0 \)

force on \( q \) in slowly varying electric field \( \vec{E} \) is

\[
\vec{F} = \int d^3r' \rho(r') \vec{E}(r')
\]

define \( \vec{r} = \vec{r}_0 + \vec{r}' \) where \( \vec{r}_0 \) is some fixed reference point in center of charge distribution \( \rho \), and \( \vec{r}' \) is distance relative to \( \vec{r}_0 \)

\[
\vec{F} = \int d^3r' \rho(r') \vec{E}(\vec{r}_0 + \vec{r}')
\]

since \( \vec{E} \) is slowly varying on length scale where \( \rho \neq 0 \), we expand

\[
\vec{F} \approx \int d^3r' \rho(r') \left[ \vec{E}(\vec{r}_0) + (\vec{r}' \cdot \vec{\nabla}) \vec{E}(\vec{r}_0) \right] + \ldots
\]

\[
= \vec{E}(\vec{r}_0) \int d^3r' \rho(r') + \left( \int d^3r' \rho(r') \vec{r}' \cdot \vec{\nabla} \right) \vec{E}(\vec{r}_0)
\]

\[
= 0 + (\phi \cdot \vec{\nabla}) \vec{E}(\vec{r}_0)
\]

\[
\vec{F} = (\phi \cdot \vec{\nabla}) \vec{E} = \sum_{\alpha=1}^{3} \phi_\alpha \frac{\partial \vec{E}}{\partial x_\alpha}
\]

For \( \vec{E} = \text{constant} \), \( \vec{F} = 0 \)
Torque on \( \mathbf{p} \) is

\[
\mathbf{N} = \int d^3r \, \mathbf{\hat{p}}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}) \approx \int d^3r \, \mathbf{\hat{p}}(\mathbf{r}) \times \left[ \mathbf{E}(\mathbf{r}) + \ldots \right]
\]

to lowest order \[ \mathbf{N} = \mathbf{\hat{p}} \times \mathbf{E} \]

Force and torque on magnetic dipoles

Localized magnetostatic current distribution \( \mathbf{j}(\mathbf{r}) \)

\[
\mathbf{F} = \frac{1}{c} \int d^3r \, \mathbf{j} \times \mathbf{B}
\]

Expand about center of current \( \mathbf{r}_0 \)

\[
\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla \mathbf{B}(\mathbf{r}_0) + \ldots
\]

\[
\mathbf{F} = \frac{1}{c} \left[ \int d^3r' \left( \mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}_0) \right) + \frac{1}{c} \int d^3r' \left( \mathbf{j}(\mathbf{r}) \times (\mathbf{r} - \mathbf{r}_0) \cdot \nabla \mathbf{B}(\mathbf{r}_0) \right) \right]
\]

from discussion of magnetic dipole approx we had \( \int d^3r \, \mathbf{j} = 0 \) for magnetostatics where \( \nabla \cdot \mathbf{j} = 0 \). So 1st term vanishes.

The 2nd term can be written as

\[
\mathbf{F}_d = \frac{\mu_0}{c} \int d^3r' \, \mathbf{j}_\mu \, r'_\perp \delta(\mathbf{r} - \mathbf{r}') \mathbf{B}_\nu
\]

For magnetostatics see magnetic dipole derivation

we need the tensor \( \int d^3r' \, \mathbf{j}_\mu \, r'_\perp \mathbf{j}_\nu = \frac{1}{c} \int d^3r' \, r'_\perp \mathbf{j}_\mu \mathbf{j}_\nu \)

\[
= \frac{1}{2c} \int d^3r' \left[ \mathbf{j}_\mu r'_\perp - r'_\perp \mathbf{j}_\mu \right]
\]

\[
= - \frac{\mu_0}{c} \mathbf{E} \mathbf{r}' \times \mathbf{f}
\]

C magnetic dipole \( \mathbf{m} = \frac{1}{2c} \int d^3r' \mathbf{r} \times \mathbf{f} \)
\[ F_\alpha = \varepsilon_{\alpha \beta \gamma} \varepsilon_{\sigma \rho \delta} (-m_\sigma) \partial_\delta B_\beta \]

\[ = -(\delta_\alpha \delta_\sigma \delta_\beta - \delta_\alpha \delta_\rho \delta_\delta) m_\sigma \partial_\delta B_\beta \]

\[ = \mu_0 \mathbf{A} \cdot \mathbf{v} (\mathbf{m} \cdot \mathbf{B}) - \mathbf{m} \cdot \mathbf{v} \mathbf{B} \]

\[ \mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B}) \]

as \( \mathbf{v} \cdot \mathbf{B} = 0 \)

**Torque on \( \mathbf{f} \)**

\[ \mathbf{N} = \frac{1}{c} \int d^3r \ \mathbf{r} \times (\mathbf{f} \times \mathbf{B}) \]

to lowest order, \( \mathbf{B} = \hat{\mathbf{B}}(\mathbf{r}) \)

\[ \mathbf{f} = \text{const over region where } \mathbf{f} \neq 0 \]

\[ = \frac{1}{c} \int d^3r \left[ \mathbf{f} \mathbf{v} (\mathbf{r} \cdot \mathbf{B}) - \mathbf{B} \mathbf{v} (\mathbf{r} \cdot \mathbf{f}) \right] \]

2nd term = 0 as follows

\[ \int d^3r \ \mathbf{r} \cdot \mathbf{f} = \int d^3r \ \mathbf{r} \cdot \mathbf{v} (\mathbf{r}^2) \] as \( \mathbf{v} (\mathbf{r}^2) = \mathbf{r} \)

\[ = - \int d^3r \ (\mathbf{r} \cdot \mathbf{f}) (\mathbf{r}^2) \]

integrate by parts.

Surface term \( \to 0 \) as \( \mathbf{f} \) is localized

as \( \mathbf{v} \cdot \mathbf{f} = 0 \) in magnetoostatics

1st term involves \( \mathbf{m} \) vector

see derivation for magnetic dipole opposite

\[ \int d^3r \ \mathbf{r} \cdot \mathbf{f} = - \int d^3r \ \mathbf{v} \cdot \mathbf{f} = \frac{1}{2} \int d^3r \left[ \mathbf{f} \mathbf{v} - \mathbf{v} \mathbf{f} \right] \]

So

\[ \mathbf{N} = \frac{1}{2c} \int d^3r \left[ \mathbf{f} \left( \mathbf{r} \cdot \mathbf{B} \right) - \mathbf{B} \left( \mathbf{r} \cdot \mathbf{f} \right) \right] \]
\[ \vec{N} = \frac{1}{2c} \int d^3r \left( \vec{\nabla} \cdot (\vec{r} \cdot \vec{B}) - \vec{r} \cdot (\vec{\nabla} \cdot \vec{B}) \right) \]

\[ \times \vec{\nabla} \times \vec{B} \]

\[ = \frac{1}{2c} \int d^3r \ (\vec{r} \times \vec{\nabla}) \times \vec{B} \]

\[ \vec{N} = \vec{m} \times \vec{B} \]
Electrostatic energy of interaction

\[ E = \frac{1}{8\pi} \int d^3r \, E^2 \]

Suppose the charge density \( \rho \) that produces \( E \) can be broken into two pieces, \( \rho = \rho_1 + \rho_2 \), with \( E = E_1 + E_2 \) where \( \nabla \cdot E_1 = 4\pi \rho_1 \) and \( \nabla \cdot E_2 = 4\pi \rho_2 \). Then

\[ E = \frac{1}{8\pi} \int d^3r \left[ E_1^2 + E_2^2 + 2E_1 \cdot E_2 \right] \]

\( \uparrow \) \( \uparrow \) \( \uparrow \)

"Self-energy" \( \uparrow \) "Self-energy" \( \uparrow \) "Interaction" energy

of \( \rho_1 \) \( \uparrow \) of \( \rho_2 \) \( \uparrow \) of \( \rho_1 \) with \( \rho_2 \)

\( E_{\text{int}} = \frac{1}{4\pi} \int d^3r \, E_1 \cdot E_2 \)

\[ = \int d^3r \, \rho_1 \Phi_2 = \int d^3r \, \rho_2 \Phi_1 \]

where \( \vec{E}_1 = -\nabla \Phi_1 \), \( \vec{E}_2 = -\nabla \Phi_2 \), by similar manipulations as earlier

\[ \text{integrals are over all space} \]

Apply to the interaction energy of a dipole in an external \( \vec{E} \) field

\( E_{\text{int}} = \int d^3r \, \rho_1 \Phi_2 \)

\( \Phi \) \text{ potential of external } \vec{E} \text{ field}

\( \rho_1 \) \text{ charge distribution of dipole}
Assuming \( \phi \) varies on length scale of \( \rho \), then we can expand \( \phi_2(\vec{r}) = \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \vec{\nabla} \phi_2(\vec{r}_0) \)

where \( \vec{r}_0 \) is the center of mass or any other convenient reference position within \( \rho \).

\[
\text{E}\text{nt} = \int d^3r \, \phi_2(\vec{r}) \left[ \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \vec{\nabla} \phi_2(\vec{r}_0) \right]
\]

\[
= q \phi_2(\vec{r}_0) + \left[ \int d^3r \, \phi_2(\vec{r})(\vec{r} - \vec{r}_0) \right] \cdot \vec{\nabla} \phi_2(\vec{r}_0)
\]

\[
= q \phi_2(\vec{r}_0) - \vec{p} \cdot \vec{E}
\]

where \( q \) is total charge in \( \rho \), and \( \vec{p} \) is dipole moment with respect to \( \vec{r}_0 \). \( \vec{E} = -\vec{\nabla} \phi_2 \) is external \( \vec{E} \)-field.

For a neutral charge distribution \( q = 0 \), and \( \vec{p} \) is independent of the origin about which it is computed, so

\[
\text{E}\text{nt} = -\vec{p} \cdot \vec{E}
\]

\( \sim \) does not include the energy needed to make the dipole or to make \( \vec{E} \).

\( \text{E}\text{nt} \) is lowest when \( \vec{p} \parallel \vec{E} \)

\( \Rightarrow \) in thermal ensemble, dipoles tend to align parallel to an applied \( \vec{E} \).
Energy of magnetic dipole in external field

We had that the force on the dipole was

\[ \vec{F} = - \nabla (m \cdot \vec{B}) \]

If we regard the force as coming from the gradient of a potential energy \( U \) then \( \vec{F} = - \nabla U \Rightarrow \)

\[ U = - m \cdot \vec{B} \]

or equivalently, energy = work done to move dipole into position from \( \vec{W} = - \int \vec{F} \cdot d\vec{l} = - \int \nabla (m \cdot \vec{B}) \cdot d\vec{l} = - m \cdot \vec{B} \)

This is the correct energy to use in cases where \( \vec{m} \) is due to intrinsic magnetic moments of atom or molecule — say from electron or nuclear spin. For a thermal ensemble magnetic moments tend to align \( \parallel \) to \( \vec{B} \).

The answer comes out quite differently if we are talking about a magnetic moment produced by a classical current loop. To see this, consider what we would get if we tried to do the calculation in a similar way to how we did if the the energy of an electric dipole in an electric field...
Magnetostatic energy of interaction

\[ \mathcal{E} = \frac{1}{\pi} \int d^3r \, \mathbf{B}^2 \]

Suppose current \( \mathbf{j} \) that produces \( \mathbf{B} \) can be divided

\[ \mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2 \]

with \( \mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 \), where

\[ \mathbf{\nabla} \times \mathbf{B}_1 = \frac{4\pi}{c} \mathbf{j}_1 \]

and

\[ \mathbf{\nabla} \times \mathbf{B}_2 = \frac{4\pi}{c} \mathbf{j}_2 \].

Then

\[ \mathcal{E} = \frac{1}{\pi} \int d^3r \left[ \mathbf{B}_1^2 + \mathbf{B}_2^2 + 2 \mathbf{B}_1 \cdot \mathbf{B}_2 \right] \]

self energy self energy interaction energy

\( \mathbf{\mathcal{E}}_{\mathcal{E}} \) \( \mathbf{\mathcal{E}}_{\mathcal{E}} \) \( \mathbf{\mathcal{E}}_{\text{interaction}} \) of \( \mathbf{j}_1 \) with \( \mathbf{j}_2 \)

\[ \mathcal{E}_{\text{int}} = \frac{1}{4\pi} \int d^3r \, \mathbf{B}_1 \cdot \mathbf{B}_2 \]

\[ = \frac{1}{c} \int d^3r \, \mathbf{j}_1 \cdot \mathbf{A}_2 = \frac{1}{c} \int d^3r \, \mathbf{j}_2 \cdot \mathbf{A}_1 \]

where \( \mathbf{\mathcal{B}}_1 = \mathbf{\nabla} \times \mathbf{A}_1 \), \( \mathbf{\mathcal{B}}_2 = \mathbf{\nabla} \times \mathbf{A}_2 \), by similar manipulations as earlier

integrals are over all space

Apply to the interaction energy of a magnetic dipole in an external \( \mathbf{B} \) field.

\[ \mathcal{E}_{\text{int}} = \frac{1}{2} \int d^3r \, \mathbf{j}_1 \cdot \mathbf{A}_2 \]

\( \mathbf{\mathcal{A}} \) vector potential of external \( \mathbf{B} \) field

\( \mathbf{\mathcal{J}} \) current distribution of dipoles
Assuming \( \hat{A} \) varies slowly on length scale of \( \frac{1}{\epsilon} \), then expand \( A_i(r) = A_i(r_0) + (r - r_0) \cdot \nabla A_i(r_0) \)

\[
E_{\text{int}} = \frac{1}{c} \int d^3r \frac{\hat{f}_i \cdot \vec{A}(r_0)}{r}
\]

\[
+ \frac{1}{c} \int d^3r \sum_i f_{ij} (r - r_0) \cdot \partial_j A_i(r_0)
\]

Stipulated to move \( \vec{A} \) new volume

From magnetostatic computation of magnetic dipole moment we had \( \int d^3r \hat{f} = 0 \)

for magnetostatics

\( \Rightarrow \) 1st term above vanishes. So does the price of 2nd term \( (\int d^3r f_{ij} \times \hat{r}) \cdot \partial_j A_i(r_0) \)

We are left with

\[
E_{\text{int}} = \left[ \frac{1}{c} \int d^3r \frac{\hat{f}_i \cdot \vec{r}_j}{r} \right] \cdot \partial_j A_i(r_0)
\]

summation over repeated indices is implied

From computation of magnetic dipole approx we had

\[
\int d^3r \hat{f}_i \times \vec{r}_j = -\int d^3r \hat{f}_{ij} \vec{r}_i
\]

Recall:

\[
\vec{m} = \frac{1}{2c} \int d^3r \hat{r} \times \hat{f} = \frac{1}{2} \int d^3r \left[ \hat{f}_{ij} \vec{r}_j - \hat{f}_{ij} \vec{r}_i \right]
\]

\[
= \frac{1}{2} \epsilon_{kij} \int d^3r \left( \hat{r} \times \hat{f} \right)_k
\]

\[
\Rightarrow \frac{1}{c} \int d^3r \hat{f}_{ij} \vec{r}_i = -\epsilon_{kij} \hat{m}_k \quad \text{mag dipole moment}
\]
\[ E_{\text{int}} = -m_k \epsilon_{kij} \partial_j A_i = m_k \epsilon_{kij} \partial_j A_i \]

\[ = \vec{m} \cdot (\nabla \times \vec{A}) = \vec{m} \cdot \vec{B} = E_{\text{int}} \]

"This is opposite in sign to what we found earlier!"

Why the difference?"

1. When we integrate the work done against the magnetostatic force to move \( \vec{m} \) into position from infinity, we found the energy

\[ U = -\vec{m} \cdot \vec{B} \]

2. When we compute the interaction energy from

\[ E_{\text{int}} = \frac{1}{2} \int d^3r \vec{A}_1 \cdot \vec{A}_2 = \frac{1}{2} \int d^3r \int d^3r \left( \frac{\vec{F}_1(\vec{r}) \cdot \vec{F}_2(\vec{r})}{|\vec{r} - \vec{r}'|} \right) \]

we found the energy \( E_{\text{int}} = +\vec{m} \cdot \vec{B} \)

To see which is correct, let us consider computing the interaction energy \( \odot \) directly via method 1.
Consider two loops with currents $I_1$ and $I_2$.

What is the work done to move loop 2 in from infinity to its final position with respect to loop 1?

Magnetostatic force on loop 2 due to loop 1 is

$$\vec{F} = \frac{I_2}{c} \oint \oint \vec{dl}_2 \times \vec{B}_1$$

Lorentz force

$$\vec{B}_1(\vec{r}) = \frac{I_1}{c} \oint \vec{dl}_1 \times (\vec{r} - \vec{r}_1)$$

Biot-Savart law

$$F = \frac{I_1 I_2}{c^2} \oint \oint \vec{dl}_2 \times \left( \frac{\vec{dl}_1 \times (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \right)$$

Use triple product rule

$$\vec{dl}_2 \times \left[ \vec{dl}_1 \times (\vec{r}_2 - \vec{r}_1) \right] = \vec{dl}_1 \left[ \vec{dl}_2 \cdot (\vec{r}_2 - \vec{r}_1) \right] - (\vec{r}_2 - \vec{r}_1)(\vec{dl}_1 \cdot \vec{dl}_2)$$

From the 1st term

$$\oint \vec{dl}_2 \cdot (\vec{r}_2 - \vec{r}_1) \left( \frac{1}{|\vec{r}_2 - \vec{r}_1|^3} \right) = 0$$

Integral of gradient around closed loop always vanishes!
\[ F = \frac{c}{2} \int_{1/2}^{1} \frac{dl}{d\nu} \left( \frac{1}{R+5\nu^2 - \nu^3} \right) \]

To make loop 2 we need to apply a magnetic force.

Thus far we have used the above magnetic interaction.

Hence, the magnitude of loop 2 depends on the current in loop 2.

Therefore, the work done in moving loop 2 from

\[ F = \frac{c}{2} \int_{1/2}^{1} \frac{dl}{d\nu} \left( \frac{1}{R+5\nu^2 - \nu^3} \right) \]
The minus sign we have here is the same minus sign we got when we found \( \mathbf{U} = -\mathbf{m} \cdot \mathbf{B} \)
by integrating the force on the magnetic dipole.

Why don't we get
\[ + \frac{1}{c^2} \int d^3r_1 d^3r_2 \frac{\mathbf{F}_1(r_1) \cdot \mathbf{F}_2(r_2)}{|r_2 - r_1|} \]
with the plus sign we expect from \( E = \frac{1}{c^4} \int d^3r B^2 \)?

Answer: we have left something out!

Faraday's Law - when we move loop 2, the magnetic
flux through loop 2 changes. This \( \frac{d\Phi}{dt} \) creates
an emf \( \oint \mathbf{dl} \cdot \mathbf{E} \) around the loop that
would tend to change the current in the loop.
If we are to keep the current fixed at constant \( I_2 \)
then there must be a battery in the loop that does
work to counter this induced emf (electromotive force).
Similarly, the flux through loop 1 is changing and a
battery does work to keep \( I_1 \) constant. We need
to add this work done by the battery to the
mechanical work computed above.

emf induced in loop 1 \( E_1 = \oint \mathbf{dl}_1 \cdot \mathbf{E}_2 \) \( \int \) integrates
emf induced in loop 2 \( E_2 = \oint \mathbf{dl}_2 \cdot \mathbf{E}_1 \) \( \int \) in direction

Faraday \( E_1 = \frac{-d\Phi_1}{c dt} \) \( \Phi_1 = \) flux through loop 1

\[ E_2 = \frac{-d\Phi_2}{c dt} \] \( \Phi_2 = \) flux through loop 2
To keep the current constant, the batteries need to provide an emf that counters these Faraday-induced emf's. The work done by the battery per unit time is therefore

\[ \frac{dW_{\text{battery}}}{dt} = -\varepsilon_1 I_1 - \varepsilon_2 I_2 \]

(check units: \( \varepsilon I \) is \([\text{length}] \cdot [\text{force/s}] = [\text{length}] \cdot [\text{force/s}] = \text{energy/s}\))

\[ \frac{dW_{\text{battery}}}{dt} = \frac{d\Phi_1}{cdt} I_1 + \frac{d\Phi_2}{cdt} I_2 \]

\[ W_{\text{battery}} = \int_0^t \left( \frac{d\Phi_1}{cdt} I_1 + \frac{d\Phi_2}{cdt} I_2 \right) dt \]

where \( t = 0 \) loop 2 is at infinity
\( t = T \) loop 2 is at final position
\( I, I \) kept constant as loop moves

\[ W_{\text{battery}} = \frac{1}{c} \Phi_1 I_1 + \frac{1}{c} \Phi_2 I_2 \]

where \( \Phi_1 \) and \( \Phi_2 \)
are fluxes in final position, and are assumed that fluxes < 0 at infinity

\[ \Phi_1 = -CM_{12} I_2 \]
\[ \Phi_2 = CM_{21} I = CM_{12} I_1 \quad \text{as} \quad M_{12} = M_{21} \]

\[ \Rightarrow W_{\text{battery}} = 2M_{12} I_1 I_2 \]
add this to the mechanical work

\[ W_{\text{total}} = W_{\text{mech}} + W_{\text{battery}} = -M_{12} I_1 I_2 + 2 M_{12} I_1 I_2 \]

\[ = M_{12} I_1 I_2 = \frac{1}{c^2} \int d^3 \vec{r}_1 \; d^3 \vec{r}_2 \; \frac{\vec{f}_1(\vec{r}_1) \cdot \vec{f}_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} \]

we get back the correct interaction energy!

**Conclusion:** The magnetostatic interaction energy

\[ \frac{1}{c^2} \int d^3 \vec{r}_1 \; d^3 \vec{r}_2 \; \frac{\vec{f}_1(\vec{r}_1) \cdot \vec{f}_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} \]

includes the work done to maintain the currents stationary as the current distributions move.

When we computed the interaction energy of a current loop dipole \( \vec{m} \) and find

\[ E_{\text{int}} = \pm \vec{m} \cdot \vec{B} \]

this includes the energy needed to maintain the constant current producing the constant \( \vec{m} \).

When we integrated the force on the dipole to find the potential energy

\[ U = -\vec{m} \cdot \vec{B} \]

this did not include the energy needed to maintain the constant current that creates \( \vec{m} \).

This is the correct energy expression to use when \( \vec{m} \) comes from intrinsic magnetic moments due to particles intrinsic spin, which cannot be viewed as arising from a current loop.