Magnetostatics

Lorentz Force

a charge q, in motion with velocity $\vec{v}$, feels the force

$$\vec{F} = q\left(\vec{E} + k_4\, \vec{v}\times\vec{B}\right) \quad \leftarrow \text{Lorentz force}$$

$\vec{B}$ is the magnetic field at the position of the charge.
k_4 is a universal constant.

Just as the constant k_1 fixed the units of charge q, the constant k_4 can be viewed as fixing the units of B magnetic field. By choosing the units of q and B appropriately, we are free to choose any values for k_1 and k_4.

Magnetic field $\vec{B}$ is generated by moving charge. A charge q' with velocity $\vec{v}'$ ($v' \ll c$) located at the origin $\vec{r}'=0$ produces a magnetic field at position $\vec{r}$,

holds only nonrelativistically $\longrightarrow$ $$\vec{B}(\vec{r}) = k_5\, q' \frac{\vec{v}'\times\vec{r}}{r^3} = \frac{k_5}{k_1}\, \vec{v}'\times\vec{E}(\vec{r})$$

k_5 is a universal constant. We will see that it cannot be chosen independently of k_1 and k_4. (since k_1 fixed units of q, and k_4 fixed units of $\vec{B}$, there are no further new quantities whose units could be adjusted to allow us to fix k_5 arbitrarily)

The force on a charge q at position $\vec{r}$, moving with velocity $\vec{v}$, due to a charge q' at the origin moving with velocity $\vec{v}'$ is, in non-relativistic limit ($v, v' \ll c$) is,

$$\vec{F} = k_1 q q' \frac{\vec{r}}{r^3} + k_4 k_5 q q' \frac{\vec{v} \times (\vec{v}' \times \vec{r})}{r^3}$$

Coulomb force — magnetic analog of Coulomb force

The magnetic part is just the point charge equivalent of the Biot-Savart law for the force between current ~~carry~~ carrying wires. If we regard $q\vec{v} = \vec{I}$ as the current of charge q, and $q'\vec{v}' = \vec{I}'$ as the current of charge q', then the magnetic force is $k_4 k_5 \, \vec{I} \times (\vec{I}' \times \frac{\vec{r}}{r^3})$ which is the Biot-Savart Law.

Rewrite above force as

$$\vec{F} = k_1 \left(1 + \frac{k_4 k_5}{k_1} \vec{v} \times \vec{v}' \times \right) \frac{\vec{r}}{r^3} q q'$$

we see that $\left(\frac{k_4 k_5}{k_1}\right)$ has units of $(\text{velocity})^{-2}$

it must be independent of whatever convention one used to choose the units of q or B (ie independent of choices for k_1 and k_4). Experimentally it is found that

$$\left(\frac{k_4 k_5}{k_1}\right) = \frac{1}{c^2}$$

c = speed of light in vacuum

Continuum current density

For charges q_i at positions $\vec{r}_i(t)$ with $\vec{v}_i = \frac{d\vec{r}_i}{dt}$
we define the current density

$$\vec{j}(\vec{r},t) = \sum_i q_i \vec{v}_i(t)\, \delta(\vec{r}-\vec{r}_i(t))$$

units of $\vec{j}$ are $(\text{charge})\left(\frac{\text{length}}{\text{time}}\right)\left(\frac{1}{\text{length}^3}\right) = \left(\frac{\text{charge}}{\text{area}\cdot\text{time}}\right)$

charge per unit area per unit time

For a surface S

$$\int_S da\, \hat{n}\cdot\vec{j} = I$$ current (charge per unit time) passing through surface S

Charge Conservation   vol V bounded by surface S

$$\frac{d}{dt}\int_V d^3r\, \rho(\vec{r},t) = -\oint_S da\, \hat{n}\cdot\vec{j}$$

rate of change of total charge in V = $(-)$ charge flowing out of V through S per unit time

use $$\oint_S da\,\hat{n}\cdot\vec{j} = \int_V d^3r\, \vec{\nabla}\cdot\vec{j} = -\int_V d^3r\, \frac{\partial}{\partial t}\rho(\vec{r},t)$$

$\Rightarrow$ local charge conservation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$

A static situation has $\frac{\partial \rho}{\partial t} = 0$

$\Rightarrow$ magnetostatics is defined by the condition $\underline{\vec{\nabla}\cdot\vec{J} = 0}$

Differential formulation of Biot-Savart

For a set of charges q_i at $\vec{r}_i$ we have

$$\vec{B}(\vec{r}) = \sum_i k_5\, q_i\, \vec{v}_i \times \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

$$= k_5 \int d^3r'\, \vec{J}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$= k_5 \int d^3r'\, \vec{J}(\vec{r}\,') \times \vec{\nabla}\left(\frac{-1}{|\vec{r}-\vec{r}\,'|}\right)$$

$$\vec{B}(\vec{r}) = k_5\, \vec{\nabla}\times\left[\int d^3r'\, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right]$$

where we used $\vec{\nabla}\times(\vec{A}\varphi) = -\vec{A}\times\vec{\nabla}\varphi$ when $\vec{A}$ is indep of $\vec{r}$

$\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ for any vector function $\vec{A}$

integral form $\oint_S da\, \hat{n}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{B} = k_5\, \vec{\nabla}\times\left[\vec{\nabla}\times\left(\int d^3r'\, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

use $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$$\vec{\nabla}\times\vec{B} = k_5\,\vec{\nabla}\left[\int d^3r'\ \vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\right]$$

$$-k_5\int d^3r'\ \vec{J}(\vec{r}')\,\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

in the 2nd term, $\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$

in the 1st term, $\vec{\nabla}\cdot\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \vec{J}(\vec{r}')\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

$$= -\vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

since $\vec{\nabla} = -\vec{\nabla}'$

So $\int d^3r'\ \vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) = -\int d^3r'\,\vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

integrate by parts $\quad = \int d^3r'\left(\vec{\nabla}'\cdot\vec{J}(r')\right)\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$
surface term $\to 0$ as
we take surface $\to\infty$
since $\vec{J}\to 0$ as $r\to\infty$

But for magnetostatics $\vec{\nabla}\cdot\vec{J} = 0 \Rightarrow$ only 2nd term remains

Thus, for magnetostatics

$$\vec{\nabla}\times\vec{B} = 4\pi k_5\,\vec{J} \qquad \text{Ampere's law}$$

integral form $\displaystyle\oint_C d\vec{l}\cdot\vec{B} = 4\pi k_5\int_S da\,\hat{n}\cdot\vec{J}$

curve bounding surface

Although above diff eqn's were derived starting from a "non-relativistic" point-charge Biot-Savart law, the actually remain true for all magnetostatic situations

So far electrostatics $\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi k_1 \rho & \text{Gauss} \\ \vec{\nabla}\times\vec{E} = 0 \end{cases}$

magnetostatics $\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} & \text{Ampere} \end{cases}$

current conservation $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

Time Dependent situations

Faraday's law of induction $\vec{\nabla}\times\vec{E} \neq 0$!

$$\oint_C d\vec{l}\cdot\vec{E} = -k_3 \frac{\partial}{\partial t}\int_S da\, \hat{n}\cdot\vec{B}$$

voltage around closed loop ~ − time rate of change of magnetic flux through loop

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -k_3 \frac{\partial \vec{B}}{\partial t}}$$ k_3 is universal constant

Maxwell correction to Ampere's law

in our derivation of $\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J}$
we used $\vec{\nabla}\cdot\vec{J} = 0$. This is only true for magnetostatics − it is NOT true in general

Alternatively, since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ always, if Amperes law was true, we would necessarily conclude that $\vec{\nabla}\cdot\vec{J} = 0$. But $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$ in general.

Proposed correction: $\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} + \vec{W}$

where $\vec{W}$ must be such that charge conservation holds.

Now

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 = 4\pi k_5 \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{W}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{W} = -4\pi k_5 \vec{\nabla}\cdot\vec{J} = 4\pi k_5 \frac{\partial \rho}{\partial t}$$ by charge conserv

$$= \frac{k_5}{k_1}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}$$ by Gauss Law

$$\Rightarrow \vec{W} = \frac{k_5}{k_1}\frac{\partial \vec{E}}{\partial t}$$

So corrected Amperes law is

$$\boxed{\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} + \frac{k_5}{k_1}\frac{\partial \vec{E}}{\partial t}}$$

Now consider $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$

$= -\nabla^2\vec{B}$ as $\vec{\nabla}\cdot\vec{B} = 0$

If there are no sources ($\rho = 0$, $\vec{J} = 0$) then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{B} = \frac{k_5}{k_1}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}$$

$$= -\frac{k_5 k_3}{k_1}\frac{\partial^2 B}{\partial t^2}$$ by Faraday

$$\nabla^2 B = \frac{k_5 k_3}{k_1}\frac{\partial^2 B}{\partial t^2}$$ this is the wave equation

$\Rightarrow \frac{k_5 k_3}{k_1}$ has units of $(\text{velocity})^{-2}$

Since we know that the above wave equation describes electromagnetic waves, ie light, then

$$\frac{k_5 k_3}{k_1} = \frac{1}{c^2}$$

we already had $\quad \dfrac{k_4 k_5}{k_1} = \dfrac{1}{c^2}$

$$\Rightarrow k_3 = k_4$$

$\Rightarrow$ k_1 and k_4 are arbitrary – they can be chosen to be anything by adjusting the units of q and B. k_3 and k_5 are then fixed by $\dfrac{k_4 k_5}{k_1} = \dfrac{1}{c^2}$, $k_3 = k_4$

Popular systems of E & M units

	k_1	$k_3 = k_4$	k_5	
MKS or SI	$\frac{1}{4\pi\epsilon_0}$	1	$\frac{\mu_0}{4\pi}$	$\left(\epsilon_0\mu_0 = \frac{1}{c^2}\right)$
Gaussian or CGS	1	$\frac{1}{c}$	$\frac{1}{c}$	
Rationalized Gaussian	$\frac{1}{4\pi}$	$\frac{1}{c}$	$\frac{1}{4\pi c}$	

In MKS, charges are measured in "coulombs"
current measured in "amps"
magnetic field measured in "tesla" = "weber/m^2"

In CGS, charges are measured in "statcoulombs"
current measured in "statamperes"
magnetic field measured in "gauss"

We will use the CGS or Gaussian units

1) $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ 3) $\vec{\nabla}\cdot\vec{B} = 0$

2) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

Maxwell's Equ's !!

$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$

Lorentz force

Physical content of Maxwell's equs.

1) Gauss Law for electric field – charge is source of $\vec{E}$ field. Field lines can begin and end at point charges

2) Faraday's Law of induction – time varying magnetic flux produces circulating $\vec{E}$-field

3) Gauss Law for magnetic fields – no magnetic monopoles. Magnetic field lines are continuous, they either close upon themselves or go off to infinity, they cannot begin nor end at any point.

4) Amperes Law + Maxwell's correction – electric current is a source for circulating $\vec{B}$-field; so is a time varying $\vec{E}$-field. Maxwell's correction is necessary to have charge conservation and to give electromagnetic waves.

Note:

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})=0$ as div of curl always vanishes

then (2) $\Rightarrow -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B})=0$

So if $\vec{\nabla}\cdot\vec{B}=0$ at $t=0$, Equ (2) requires that $\vec{\nabla}\cdot\vec{B}$ remains zero for all time

Similarly

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})=0$

then (4) $\Rightarrow \frac{4\pi}{c}\vec{\nabla}\cdot\vec{j}+\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})=0$

use charge continuity $\vec{\nabla}\cdot\vec{j}=-\frac{\partial\rho}{\partial t}$ to get

$$-\frac{4\pi}{c}\frac{\partial\rho}{\partial t}+\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})=0$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}-4\pi\rho)=0$$

So if $\vec{\nabla}\cdot\vec{E}=4\pi\rho$ at $t=0$, Equ (4) requires that $\vec{\nabla}\cdot\vec{E}=4\pi\rho$ for all time

Thus Equs (1) and (3) can be viewed as "initial conditions". If they are true for any particular moment in time, then Equs (2) and (4) ensure that they remain true for all time.

Equs (2) and (3) are also refered to as the homogeneous Maxwell Equ's – they involve only the fields $\vec{E}$ & $\vec{B}$ and not the sources ρ and $\vec{j}$. Eqs (1) and (4) are refered to as the inhomogeneous Maxwell's equs – they involve the sources ρ and $\vec{j}$.