EM waves in Conductors

conduction electrons are mobil, not bound
$\Rightarrow$ we have to include the $\vec{J}_f$ and ρ_f from them.

Simple classical model for electron motion – "Drude Model"

$$m\ddot{\vec{r}} = \underbrace{-e\vec{E}(t)}_{\text{external } E \text{ field}} - \underbrace{\frac{m}{\tau}\dot{\vec{r}}}_{\text{damping force due to collisions}}$$

τ is "relaxation time"

$$\vec{E} = \vec{E}_\omega e^{-i\omega t} \Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t} \text{ solution}$$

plug in to get

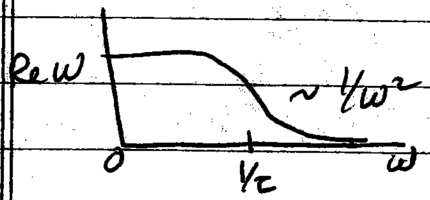
$$\left(-\omega^2 - \frac{i\omega}{\tau}\right)\vec{r}_\omega = -\frac{e}{m}\vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m}\frac{1}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega$$

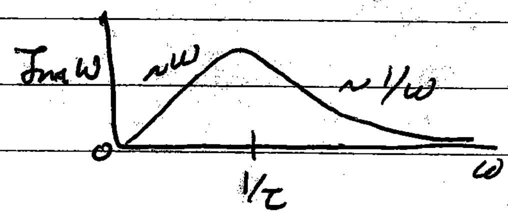
current is $\vec{J}_f = -e\underbrace{n}_{\text{density of electrons}}\dot{\vec{r}}_\omega = -en(-i\omega)\vec{r}_\omega$

$$\vec{J}_f = \frac{ne^2}{m}\frac{i\omega}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}\vec{E}_\omega$$

$$\vec{J}_f = \sigma(\omega)\,\vec{E}_\omega$$

conductivity $\boxed{\sigma(\omega) = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}}$





$$\mathrm{Re}\,\sigma = \frac{\sigma_0}{1+\omega^2\tau^2} \qquad \mathrm{Im}\,\sigma = \frac{\sigma_0\,\omega\tau}{1+\omega^2\tau^2}$$

$$\sigma_0 = \sigma(0) = \frac{ne^2\tau}{m} \quad \text{dc conductivity}$$

Charge density ρ_f given by charge conservation law.
for plane waves

$$\rho_f = \rho_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \vec{j}_f = \vec{j}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{j}_f \Rightarrow -i\omega\rho_\omega = -i\vec{k}\cdot\vec{j}_\omega$$

$$\rho_\omega = \frac{\vec{k}\cdot\vec{j}_\omega}{\omega} = \frac{\sigma(\omega)\,\vec{k}\cdot\vec{E}_\omega}{\omega}$$

<u>Maxwell Equations</u>

1) $\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$ 2) $\vec{\nabla}\cdot\vec{B} = 0$

3) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j}_f + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$

Assume $\vec{H} = \vec{B}/\mu$, μ constant

$\vec{D}_\omega = \varepsilon_b(\omega)\vec{E}_\omega$ $\varepsilon_b(\omega)$ is dielectric function from the bound charges

$\vec{j}_\omega = \sigma(\omega)\vec{E}_\omega$

$\rho_\omega = \frac{\sigma}{\omega}\vec{k}\cdot\vec{E}_\omega$ $\sigma(\omega)$ is conductivity from free charges

For harmonic plane wave solutions $\vec{E} = E_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ etc.

1) $\Rightarrow \quad i\vec{k}\cdot\vec{D}_\omega = i\vec{k}\cdot\varepsilon_b E_\omega = 4\pi\rho_\omega = 4\pi\sigma\dfrac{\vec{k}\cdot\vec{E}_\omega}{\omega}$

$\Rightarrow \quad i\vec{k}\cdot\vec{E}_\omega\left(\varepsilon_b + \dfrac{4\pi i\sigma}{\omega}\right) = 0$

2) $\Rightarrow \quad i\mu\vec{k}\cdot\vec{H}_\omega = 0$

3) $\Rightarrow \quad i\vec{k}\times\vec{E}_\omega = \dfrac{i\omega\vec{B}_\omega}{c} = \dfrac{i\omega\mu}{c}H_\omega$

4) $\Rightarrow \quad i\vec{k}\times H_\omega = \dfrac{4\pi}{c}\vec{j}_\omega - \dfrac{i\omega}{c}D_\omega$

$$= \frac{4\pi}{c}\sigma\vec{E}_\omega - \frac{i\omega}{c}\varepsilon_b\vec{E}_\omega$$

$$= -\frac{i\omega}{c}\left(\varepsilon_b + \frac{4\pi i\sigma}{\omega}\right)\vec{E}_\omega$$

Notice: all the equations above look exactly like what we had for the dielectric, provided we define

$$\boxed{\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{4\pi i\sigma(\omega)}{\omega}}$$

So all results for the dielectric case carry over to conductors, provided we make the above substitution. In particular

dispersion relation for transverse modes $\boxed{k^2 = \dfrac{\omega^2}{c^2}\mu\,\varepsilon(\omega)}$

The main difference between dielectrics & conductors has to do with the contribution that the $4\pi i\sigma/\omega$ makes to the real and imaginary parts of $\varepsilon(\omega)$.

For simple Drude model $\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau}$ $\sigma_0 = \frac{ne^2\tau}{m}$

① Low frequencies $\omega \ll 1/\tau$

$\varepsilon_b(\omega) \approx \varepsilon_b(0)$ real

$\sigma(\omega) \approx \sigma_0$ real

$\Rightarrow \boxed{\varepsilon(\omega) \simeq \varepsilon_b(0) + \frac{4\pi i\sigma_0}{\omega}}$ ← gives large ε_2 as $\omega \to 0$

⇒ strong dissipation

$\mathrm{Re}\,\varepsilon = \varepsilon_1$

$\mathrm{Im}\,\varepsilon = \varepsilon_2$

when $\frac{\varepsilon_2}{\varepsilon_1} = \frac{4\pi\sigma_0}{\omega\varepsilon_b(0)} \gg 1$ we call this regime a "good" conductor.
conduction electrons dominate the response
– waves strongly attenuated

when $\frac{\varepsilon_2}{\varepsilon_1} = \frac{4\pi\sigma_0}{\omega\,\varepsilon_b(0)} \ll 1$ we call this regime a "poor" conductor.
little absorption of energy by conduction electrons.
waves propagate

one always enters the "good" conductor region when ω gets sufficiently small.

wave vector:

$$k = \frac{\omega}{c}\sqrt{\mu \varepsilon}$$

for a good conductor where $\varepsilon_2 \gg \varepsilon_1$,

$$\varepsilon \sim i\varepsilon_2 = \frac{4\pi i \sigma_0}{\omega}$$

$$k = k_1 + i k_2 = \frac{\omega}{c}\sqrt{\mu \frac{4\pi i \sigma_0}{\omega}} \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$k_1 = k_2 = \frac{\omega}{c}\sqrt{\frac{4\pi\mu\sigma_0}{2\omega}} = \frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

for $\vec{k} = k\hat{z}$, $\quad \vec{E} = \vec{E}_\omega e^{i(kz-\omega t)} = \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$

$$\delta \equiv 1/k_2 = \frac{c}{\sqrt{2\pi\mu\sigma_0\omega}}$$

"skin depth" distance wave propagates into conductor

$\delta \sim 1/\sqrt{\omega}$ increases as ω decreases

ϕ phase shift between oscillations of $\vec{E}$ and $\vec{H}$

$$\phi = \arctan(k_2/k_1) \simeq \arctan(1) = 45^\circ$$

Amplitude ratio
$$\frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{c|k|}{\omega\mu} = \frac{\sqrt{2}\,c}{\omega\mu}k_1$$

$$= \frac{\sqrt{2}\,c}{\omega\mu}\frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

$$= \sqrt{\frac{4\pi\sigma_0}{\omega\mu}} \sim 1/\sqrt{\omega}$$

as $\omega \to 0$, most of the energy of the wave is carried by the magnetic field part

② high frequencies $\omega \gg 1/\tau$, $\omega \gg \omega_0$

$$\varepsilon_b(\omega) \approx 1$$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{i n e^2 \tau}{m\omega\tau} = \frac{i n e^2}{m\omega}$$ pure imaginary, indep of τ

$$\varepsilon(\omega) \approx 1 + \frac{4\pi i \sigma}{\omega} \approx 1 - \frac{4\pi n e^2}{m\omega^2}$$

$$\boxed{\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}}$$ $\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$ plasma freq of the conduction electrons

$\varepsilon(\omega)$ is <u>real</u>

1) If $\omega > \omega_p$ then $\varepsilon > 0$

$\Rightarrow$ transparent propagation

$k = k_1 = \frac{\omega}{c}\sqrt{\mu\varepsilon}$ is pure real

$k_2 \approx 0$

2) If $\omega < \omega_p$ then $\varepsilon < 0$

$\Rightarrow$ total reflection

$k_1 \approx 0$

$k = k_2 = \frac{\omega}{c}\sqrt{\mu|\varepsilon|}$ k is pure imaginary

ω_p gives cross over between total reflection and transparent propagation for typical metals

$$\tau \sim 10^{-14}\,\text{sec}$$

$$\omega_p \sim 10^{16}\,\text{sec}^{-1}$$

$$\lambda_p = \frac{2\pi c}{\omega_p} \sim 3\times10^{3}\,\text{Å} \qquad (\text{visible is } \lambda \sim 5\times10^{3}\,\text{Å})$$

<u>Example</u>: The ionesphere is a layer of charged gas surrounding the earth. In many respects the charged particles of the ionesphere behave like conduction electrons in a metal. The plasma freq of the ionesphere is such that

for AM radio $\omega_{AM} < \omega_p \Rightarrow$ AM radio signals reflected back to earth

for FM radio $\omega_{FM} > \omega_p \Rightarrow$ FM radio signals propagate through ionesphere into space

Explains why you can pick up AM stations from far away - they get reflected back

But you can only pick up local FM stations.

Longitudinal modes in conductors

ie $\vec{H}_\omega$ or $\vec{E}_\omega$ not $\perp \vec{k}$

magnetic field

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow i\mu\vec{k}\cdot\vec{H}_\omega = 0 \Rightarrow \vec{H}_\omega \perp \vec{k}$ transverse

or $\vec{k} = 0$ spatially uniform $\vec{H}$

if $\vec{k} = 0$ then Faraday

$$\underbrace{i\vec{k}\times\vec{E}_\omega}_{0 \text{ as } \vec{k}=0} = \frac{i\omega\mu}{c}\vec{H}_\omega = 0 \Rightarrow \omega = 0$$

So only possible longitudinal $\vec{H}$ is spatially uniform, constant in time.

electric field

$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f \Rightarrow i\epsilon(\omega)\vec{k}\cdot\vec{E}_\omega = 0 \Rightarrow \vec{E}_\omega \perp \vec{k}$ transverse

or $\epsilon(\omega) = 0$

If $\vec{E}_\omega \parallel \vec{k}$ but $\epsilon(\omega) = 0$, then can satisfy all other maxwell equations.

$$i\vec{k}\times\vec{E}_\omega = \frac{i\omega\mu}{c}\vec{H}_\omega \Rightarrow \vec{H}_\omega = 0$$

$$\Rightarrow i\mu_0\vec{k}\cdot\vec{H}_\omega = 0 \quad \text{and} \quad \underbrace{i\vec{k}\times\vec{H}_\omega}_{0 \text{ as } \vec{H}_\omega = 0} = \underbrace{-\frac{i\omega}{c}\epsilon(\omega)\vec{E}_\omega}_{0 \text{ as } \epsilon(\omega)=0}$$

So we can have longitudinal electric field oscillation when $\epsilon(\omega) = 0$

low freq $\quad \omega \ll \omega_0 \quad \omega\tau \ll 1$

$$\varepsilon \simeq \varepsilon_b(0) + \frac{4\pi i \sigma_0}{\omega}$$

$$\varepsilon(\omega) = 0 \quad \text{when} \quad \omega = -\frac{4\pi i \sigma_0}{\varepsilon_b(0)}$$

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_\omega e^{-\frac{4\pi\sigma_0}{\varepsilon_b(0)}t} e^{i\vec{k}\cdot\vec{r}}$$

If set up a longitudinal $\vec{E}$ field, it decays to zero exponentially with ~~time const~~ decay time $\varepsilon_b(0)/4\pi\sigma_0$
This is consistant with assumption the $\vec{E}=0$ inside a conductor for electrostatics

in statics $\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{E} \sim -i\vec{k}\phi_k e^{i\vec{k}\cdot\vec{r}}$ is longitudinal

high freq $\quad \omega \gg 1/\tau, \quad \omega \gg \omega_0$

$$\varepsilon(\omega) \approx 1 + \frac{4\pi i\sigma}{\omega} = 1 - \frac{\omega_p^2}{\omega^2} \qquad \omega_p^2 = \frac{4\pi n e^2}{m}$$

$\varepsilon = 0$ when $\omega = \omega_p$

So we have oscillatory longitudinal $\vec{E}$ only when $\omega = \omega_p$, independent of $\vec{k}$.

$$\vec{E} = \vec{E}_\omega e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_p t}$$

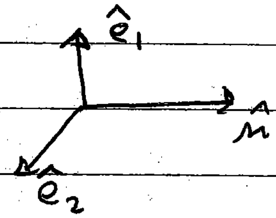
This is called a plasma oscillation. When one quantizes this oscillatory mode, it is called a plasmon

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \rho = \frac{i\vec{k}\cdot\vec{E}_\omega}{4\pi} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_p t}$ $\leftarrow$ plasma osc. is a charge density oscillation

Polarization

Consider transverse wave propagating in direction $\hat{n}$

ie $\vec{k} = k\hat{n}$.



$$\hat{e}_1 \times \hat{e}_2 = \hat{n}$$
$$\hat{n} \times \hat{e}_1 = \hat{e}_2$$
$$\hat{e}_2 \times \hat{n} = \hat{e}_1$$

A general solution for a transverse wave has the form

$$\vec{E}(\vec{r},t) = Re\left\{ (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

$$\vec{H}(\vec{r},t) = \frac{c}{\omega\mu} Re\left\{ k\, \hat{n} \times (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

$$= \frac{c}{\mu\omega} Re\left\{ k (E_1 \hat{e}_2 - E_2 \hat{e}_1) e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

So far we considered implicitly only the case

E_1, E_2 real constants $\rightarrow |\vec{E}_\omega| = \sqrt{E_1^2 + E_2^2}$

But most general case is

$$E_1 = E \cos\theta \, e^{i\chi_1}$$
$$E_2 = E \sin\theta \, e^{i\chi_2}$$

need not be real, can be complex with relative phase difference

$$E^2 = |E_1|^2 + |E_2|^2$$

(for $\chi_1 = \chi_2 = 0$, θ is angle $\vec{E}_\omega$ makes with respect to $\hat{e}_1$)

define $\Phi = k_1 \hat{n} \cdot \vec{r} - \omega t$

$\tan \varphi = k_2 / k_1$

$$\vec{E} = E\, e^{-k_2 \hat{n}\cdot\vec{r}} \left[\hat{e}_1 \cos\theta \cos(\Phi + \chi_1) + \hat{e}_2 \sin\theta \cos(\Phi + \chi_2)\right]$$

$$\vec{H} = \frac{c|k|}{\omega\mu} E\, e^{-k_2 \hat{n}\cdot\vec{r}} \left[\hat{e}_2 \cos\theta \cos(\Phi + \chi_1 + \varphi) - \hat{e}_1 \sin\theta \cos(\Phi + \chi_2 + \varphi)\right]$$

special cases

1) $\chi_1 = \chi_2$

$$\vec{E} = (\hat{e}_1 \cos\theta + \hat{e}_2 \sin\theta) E\, e^{-k_2 \hat{n}\cdot\vec{r}} \cos(\Phi + \chi_1)$$
$$\vec{H} = (-e_1 \sin\theta\ \hat{e}_2 \cos\theta) E\, e^{-k_2 \hat{n}\cdot\vec{r}} \cos(\Phi + \chi_1 + \varphi)$$

Linearly polarized - $\vec{E}$ and $\vec{H}$ point in fixed directions and are orthogonal. phase shift is φ

2) $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \pm 1/\sqrt{2}$, $\chi_1 = 0$, $\chi_2 = \chi$ (can always choose $\chi_1 = 0$ by shifting the time scale)

$$\vec{E}^{\pm} = \frac{E}{\sqrt{2}}\left(\hat{e}_1 \cos\Phi \pm \hat{e}_2 \cos(\Phi + \chi)\right)$$

Find locus of points that $\vec{E}$ sweeps out as Φ varies.

$$\vec{E}^{\pm} \cdot \hat{e}_1 = E_1^{\pm} = \frac{E}{\sqrt{2}} \cos\Phi$$
$$\vec{E}^{\pm} \cdot \hat{e}_2 = E_2^{\pm} = \pm \frac{E}{\sqrt{2}} \cos(\Phi + \chi)$$

Define $\Theta = \Phi + \chi/2$ so

$$E_1^{\pm} = \frac{E}{\sqrt{2}} \cos(\Theta - \chi/2)$$
$$E_2^{\pm} = \pm\frac{E}{\sqrt{2}} \cos(\Theta + \chi/2)$$

$$E_1^{\pm} = \frac{E}{\sqrt{2}} \cos\Theta \cos\chi/2 + \frac{E}{\sqrt{2}} \sin\Theta \sin\chi/2$$
$$E_2^{\pm} = \pm\frac{E}{\sqrt{2}} \cos\Theta \cos\chi/2 \mp \frac{E}{\sqrt{2}} \sin\Theta \sin\chi/2$$

$$\Rightarrow \quad E_1^+ + E_2^+ = \frac{2E}{\sqrt{2}} \cos\Phi \cos\chi/2$$

$$E_1^+ - E_2^+ = \frac{2E}{\sqrt{2}} \sin\Phi \sin\chi/2$$

$$\Rightarrow \quad \frac{(E_1^+ + E_2^+)^2}{2E^2\cos^2\chi/2} + \frac{(E_1^+ - E_2^+)^2}{2E^2\sin^2\chi/2} = \cos^2\Phi + \sin^2\Phi = 1$$

Similarly

$$\frac{(E_1^- - E_2^-)^2}{2E^2\cos^2\chi/2} + \frac{(E_1^- + E_2^-)^2}{2E^2\sin^2\chi/2} = 1$$

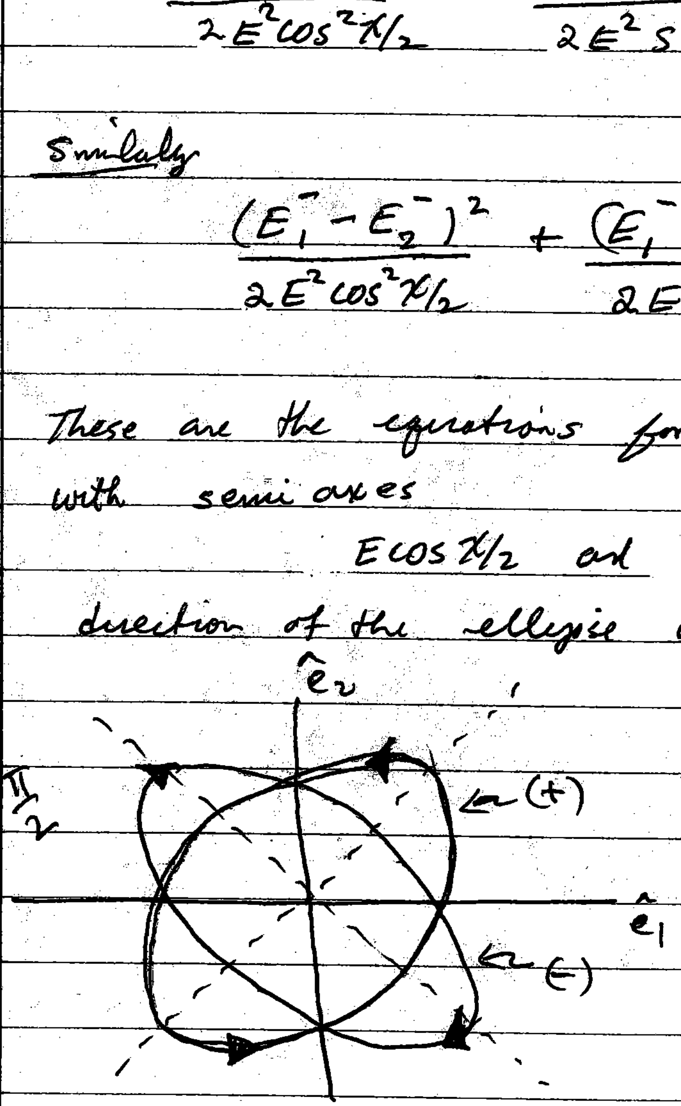
These are the equations for ellipses! $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

with semi axes

$E\cos\chi/2$ and $E\sin\chi/2$ $\qquad \hat{e}_1' \qquad \hat{e}_2'$

direction of the ellipse axes are $\left(\frac{\hat{e}_1 + \hat{e}_2}{\sqrt{2}}\right)$ and $\left(\frac{\hat{e}_1 - \hat{e}_2}{\sqrt{2}}\right)$

for $|\chi| < \pi/2$



$\vec{E}\cdot\hat{e}_1' = \frac{E_1 + E_2}{\sqrt{2}}$, $\vec{E}\cdot\hat{e}_2' = \frac{E_1 - E_2}{\sqrt{2}}$

so $|\vec{E}|$ sweeps out ellipse as Φ varies.

$\Rightarrow$ sit at position $\vec{r}$. as t varies, $|\vec{E}|$ sweeps out ellipse axis at 45° to $\hat{e}_1, \hat{e}_2$

elliptically polarized wave

For $0 < \chi < \pi/2$

For (+) $\vec{E}$ moves around ellipse counterclockwise (right handed)

For (−) $\vec{E}$ moves around ellipse clockwise (left handed)

as t increases (ie as Φ decreases)

3) <u>special case of 1c) $\chi = \pi/2$</u>

$\cos^2 \chi/2 = \sin^2 \chi/2 = 1$ ellipse axes are equal!

$\Rightarrow \vec{E}$ sweeps out a circular path

<u>circularly polarized waves</u>

(+) goes counterclockwise

(−) goes clockwise

One defines <u>circular polarization basis</u> vectors as:

$$\hat{e}_\pm \equiv \frac{1}{\sqrt{2}} (\hat{e}_1 \pm i \hat{e}_2)$$

$$\Rightarrow \quad \hat{e}_\pm^* \cdot \hat{e}_\pm = \hat{e}_\pm \cdot \hat{e}_\mp = 1$$

$$\hat{e}_\pm \cdot \hat{e}_\mp^* = \hat{e}_\pm \cdot \hat{e}_\pm = 0$$

$$\hat{e}_\pm \cdot \hat{n} = 0$$

$$\hat{n} \times \hat{e}_\pm = i \hat{e}_\pm^*$$

with this notation, a circularly polarized wave is

$$\vec{E} = Re \left\{ E \hat{e}_\pm e^{i(k \cdot r - \omega t)} \right\}$$

(+) is counterclockwise

(−) is clockwise

with $\hat{n}$ pointing <u>out</u>

Note:

$$\frac{1}{\sqrt{2}} \left(E \hat{e}_+ + E \hat{e}_- \right) = \frac{E}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \hat{e}_1 + \frac{i}{\sqrt{2}} \hat{e}_2 + \frac{1}{\sqrt{2}} \hat{e}_1 - \frac{i}{\sqrt{2}} \hat{e}_2 \right)$$

$$= E \hat{e}_1$$

and $\frac{1}{\sqrt{2}} (E \hat{e}_+ - E \hat{e}_-) = E \hat{e}_2$

Thus a linearly polarized wave can be written as a superposition of counter rotating circularly polarized waves!

general case $\quad E_1, E_2$ complex

write $E_1\hat{e}_1 + E_2\hat{e}_2 \equiv \vec{u}\, e^{i\psi}$

where ψ is chosen so that $\vec{u}\cdot\vec{u}$ is real

(can always do this since $\vec{u}\cdot\vec{u} = (E_1^2+E_2^2)\,e^{-2i\psi}$

so 2ψ is just the phase of $E_1^2+E_2^2$)

$\vec{u}$ is complex vector $\Rightarrow \vec{u} = \vec{u}_a - i\vec{u}_b$, $\vec{u}_a, \vec{u}_b$ real

since $\vec{u}\cdot\vec{u}$ is real $\Rightarrow \vec{u}_a\cdot\vec{u}_b = 0$

so $\vec{u}_a \perp \vec{u}_b$ orthogonal

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\vec{u}\, e^{i\psi}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} \qquad \Phi \equiv k_1\hat{n}\cdot\vec{r} - \omega t$$

$$= e^{-k_2\hat{n}\cdot\vec{r}}\left\{\vec{u}_a\cos(\Phi+\psi) + \vec{u}_b\sin(\Phi+\psi)\right\}$$

Define E_a as component of $\vec{E}$ in direction of $\vec{u}_a$

E_b as component of $\vec{E}$ in direction of $\vec{u}_b$

$$E_a = u_a\cos(\Phi+\psi)$$
$$E_b = u_b\sin(\Phi+\psi)$$

(ignore attenuation factor by either absorbing it into u_a, u_b, or consider $\vec{r}=0$)

$$\Rightarrow \left(\frac{E_a}{u_a}\right)^2 + \left(\frac{E_b}{u_b}\right)^2 = 1$$

elliptical polarization

$u_a \equiv |\vec{u}_a|$
$u_b \equiv |\vec{u}_b|$

semi axes of lengths u_a and u_b oriented in directions $\vec{u}_a$ and $\vec{u}_b$

if $u_a = \pm u_b$ then circularly polarized

Define circular polarization basis vectors

$$\hat{e}_\pm = \frac{1}{\sqrt{2}}(\hat{e}_a \pm i\hat{e}_b) \qquad \hat{e}_a = \frac{\vec{u}_a}{|\vec{u}_a|}, \quad e_b = \frac{\vec{u}_b}{|\vec{u}_b|}$$

any general $\vec{u}$ can always be written as:

$$\vec{u} = \frac{1}{\sqrt{2}}(u_a + u_b)\,\hat{e}_- + \frac{1}{\sqrt{2}}(u_a - u_b)\,\hat{e}_+$$

Thus a general elliptically polarized wave can be written as a superposition of circularly polarized waves!

<u>Consider behavior of magnetic field</u>

for $\chi_1 = 0$, $\chi_2 = \chi$

$$\vec{E}\cdot\vec{H} = \frac{c|k|}{\omega\mu}E^2 e^{-2k_2\hat{n}\cdot\vec{r}}\cos\theta\sin\theta\left[\cos(\Phi+\chi)\cos(\Phi+\varphi) - \cos(\Phi)\cos(\Phi+\chi+\varphi)\right]$$

can show that $[\cdots] = \sin\varphi\sin\chi$

use your trig identities!

$\Rightarrow$ $\vec{H} \perp \vec{E}$ when (i) $\chi = 0$ ie linear polarization for any φ, ie any dissipation

or (ii) $\varphi = 0$, ie no dissipation for any χ, ie any non-linear polarization

$\vec{E}\cdot\vec{H}$ is independent of Φ

$\Rightarrow$ it is constant in time and space

for elliptically polarized wave in dissipative medium, $\vec{E}\cdot\vec{H} \neq 0$