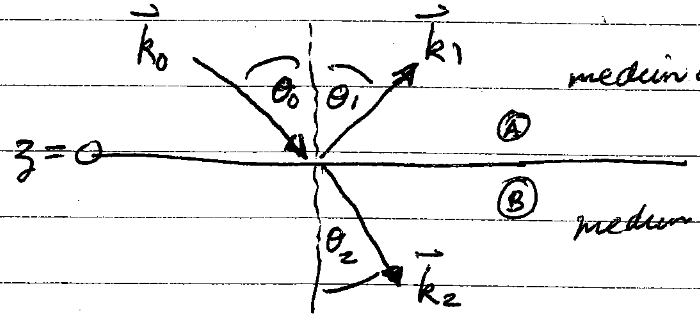
Reflection + Transmission of Waves at Interfaces



consider wave propagating from medium A into medium B.

for simplicity assume ϵ_a is real and positive, ϵ_b may be complex
μ_a and μ_b are real and constant

$\vec{k}_0$ is incident wave, θ_0 = angle of incidence
$\vec{k}_1$ is reflected wave, θ_1 = angle of reflection
$\vec{k}_2$ is the transmitted or "refracted" wave, θ_2 = angle of refraction

Let each wave be given by

$$\vec{F}_n(\vec{r},t) = \vec{F}_n\, e^{i(\vec{k}_n\cdot\vec{r} - \omega_n t)}$$

where $\vec{F}_n$ can be either $\vec{E}_n$ or $\vec{H}_n$ for the electric or magnetic component of the wave

boundary condition: tangential component $\vec{E}$ must be continuous at $z=0$. If $\hat{t}$ is a vector in xy plane, and we consider $\vec{r}=0$, then

$$\Rightarrow \quad \hat{t}\cdot\vec{E}_0\, e^{-i\omega_0 t} + \hat{t}\cdot\vec{E}_1\, e^{-i\omega_1 t} = \hat{t}\cdot\vec{E}_2\, e^{-i\omega_2 t}$$

must be true for all time. Can only happen if

$$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega}$$ all frequencies are equal

Now consider the same boundary condition for $\vec{\rho}$ a position vector in the xy plane at $z=0$. Since ω's all equal we can cancel out the common $e^{-i\omega t}$ factors to get

$$\hat{t}\cdot\vec{E}_0 e^{i\vec{k}_0\cdot\vec{\rho}} + \hat{t}\cdot E_1 e^{i\vec{k}_1\cdot\vec{\rho}} = \hat{t}\cdot\vec{E}_2 e^{i\vec{k}_2\cdot\vec{\rho}}$$

this must be true for all $\vec{\rho}$. Can only happen if the projections of the $\vec{k}_n$ in the xy plane are all equal

$$\boxed{\begin{matrix} k_{0x} = k_{1x} = k_{2x} \\ k_{0y} = k_{1y} = k_{2y} \end{matrix}}$$

only z components $\vec{k}$ vectors can be different

choose coord system as in diagram so that all $\vec{k}$ vectors lie in the xz plane (y is out of page)

Since ϵ_a is real and positive, $|\vec{k}_0|$ and $|\vec{k}_1|$ are real

$$k_{0x} = k_{1x} \Rightarrow |\vec{k}_0| \sin\theta_0 = |\vec{k}_1| \sin\theta_1$$

since $k_0^2 = \frac{\omega^2}{c^2}\sqrt{\mu_a\epsilon_a}$ and $k_1^2 = \frac{\omega^2}{c^2}\sqrt{\mu_a\epsilon_a}$

then $|\vec{k}_0| = |\vec{k}_1|$ so $\sin\theta_0 = \sin\theta_1$

$$\boxed{\theta_0 = \theta_1}$$

angle of incidence = angle of reflection

If ϵ_b is also real and positive (B is transparent) then $|\vec{k}_2|$ is real

$$k_{0x} = k_{2x} \Rightarrow |\vec{k}_0| \sin\theta_0 = |\vec{k}_2| \sin\theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2}\sqrt{\mu_b \epsilon_b}$$

$$\Rightarrow \sqrt{\mu_a \epsilon_a}\, \sin\theta_0 = \sqrt{\mu_b \epsilon_b}\, \sin\theta_2$$

in terms of index of refraction $n = \frac{kc}{\omega} = \frac{\omega\sqrt{\mu\epsilon}}{c}\frac{c}{\omega}$

$$n = \sqrt{\mu\epsilon}$$

$$\Rightarrow n_a \sin\theta_0 = n_b \sin\theta_2$$

$$\boxed{\frac{\sin\theta_2}{\sin\theta_0} = \frac{n_a}{n_b}}$$

Snell's Law
true for all types of waves, not just EM waves

If $n_a > n_b$ then $\theta_2 > \theta_0$
In this case, when θ_0 is too large, we will have
$\frac{n_a}{n_b}\sin\theta_0 > 1$ and there will be no solution for θ_2
$\Rightarrow$ no transmitted wave

This is "total internal reflection" – wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

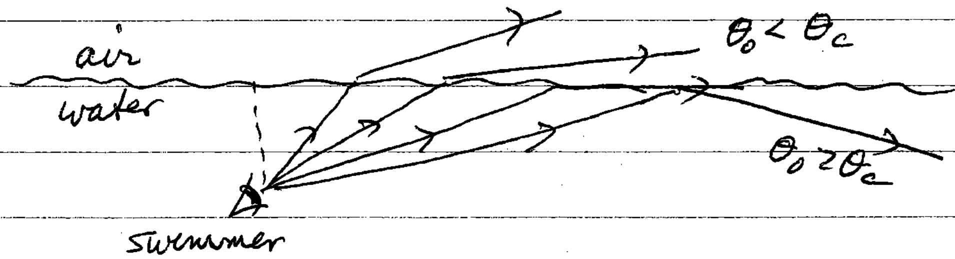
$$\frac{n_a}{n_b}\sin\theta_c = 1 \quad , \quad \boxed{\theta_c = \arcsin\left(\frac{n_b}{n_a}\right)}$$

$\epsilon \sim 1 + 4\pi N \alpha$ ← density

since $n = \sqrt{\mu\epsilon}$ and ϵ grows with density of the material, one usually has total internal reflection when one goes from a denser to a less dense medium.

Examples: diamonds sparkle due to total internal reflection. Diamonds have large $n \Rightarrow$ small θ_c $\Rightarrow$ light bounces around inside many times before it can exit

Can also see total internal reflection when swimming under water



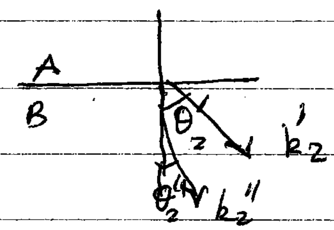
More general case $\sqrt{\epsilon_b}$ is complex so $\vec{k}_2$ is complex

$$\vec{k}_2 = \underset{\text{real part}}{\vec{k}_2'} + i\underset{\text{imaginary part}}{\vec{k}_2''} \qquad k_2' = |\vec{k}_2'|, \quad k_2'' = |\vec{k}_2''|$$

Note $\vec{k}_2'$ and $\vec{k}_2''$ need not be in the same direction!

Condition $k_{0x} = k_{2x} \Rightarrow \begin{cases} k_{0x} = k_{2x}' \\ 0 = k_{2x}'' \end{cases}$ equate real and imaginary parts

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$
$$0 = k_2'' \sin\theta_2''$$



$\Rightarrow \theta_2'' = 0$ ⎤ attenuation factor for the transmitted
$\vec{k}_2'' = k_2'' \hat{z}$ ⎦ wave is $e^{-k_2'' z}$

$\rightarrow$ planes of constant <u>amplitude</u> are <u>parallel</u> to the interface no matter what the angle of incidence θ_0

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$

$$k_0 = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a} = \frac{\omega}{c} n_a$$

planes of <u>constant phase</u> are $\perp$ to $\vec{k}_2'$

dispersion relation

$$k_2^2 = \vec{k}_2 \cdot \vec{k}_2 = (k_2')^2 - (k_2'')^2 + 2i\,\vec{k}_2' \cdot \vec{k}_2'' = \frac{\omega^2}{c^2}\mu_b \epsilon_b$$

$$\vec{k}_2' \cdot \vec{k}_2'' = k_2' k_2'' \cos\theta_2'$$

equate real and imaginary parts

$$(k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$\epsilon_b = \epsilon_{b1} + i\epsilon_{b2}$$

real

$$2k_2' k_2'' \cos\theta_2' = \frac{\omega^2}{c^2}\mu_b \epsilon_{b2}$$

Solve

$$(k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$(k_2')^2 = \left(\frac{\omega^2}{c^2}\frac{\mu_b \epsilon_{b2}}{2k_2' \cos\theta_2'}\right)^2 + \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$(k_2')^4 - \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}(k_2')^2 - \frac{\omega^4}{c^4}\frac{\mu_b^2 \epsilon_{b2}^2}{4\cos^2\theta_2'} = 0$$

quadratic formula

$$(k_2')^2 = \frac{\omega^2 \mu_b \epsilon_{b1}}{c^2\, 2} + \sqrt{\frac{\omega^4 \mu_b^2 \epsilon_{b1}^2}{c^2\, 4} + \frac{\omega^4 \mu_b^2 \epsilon_{b2}^2}{c^2\, 4\cos^2\theta_2'}}$$

$$\boxed{k_2' = \frac{\omega}{c}\sqrt{\mu_b}\left[\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

and

$$(k_2'')^2 = (k_2')^2 - \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$\boxed{k_2'' = \frac{\omega}{c}\sqrt{\mu_b}\left[-\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

Note, these reduce to what we had earlier for a plane wave, if we take $\theta_2' = 0$

Both k_2' and k_2'' depend on angle of refraction θ_2'

Finally: $k_2' \sin\theta_2' = \frac{\omega}{c} n_a \sin\theta_0$

$$\Rightarrow \boxed{n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\epsilon_{b2}^2}{\epsilon_{b1}^2\cos^2\theta_2'}}\right]^{1/2}\sin\theta_2'}$$

determines θ_2' in terms of given θ_0

Cases

(1) for a nearly transparent material with $\epsilon_{b2} \ll \epsilon_{b1}$

define $n_b = \sqrt{\mu_b \epsilon_{b1}}$ index of refraction

$$n_a \sin\theta_0 = n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{4\epsilon_{b1}^2 \cos^2\theta_2'}\right]^{1/2}$$

$$\cong n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \cos^2\theta_2'}\right]$$

small correction to Snell's law

for $\frac{\epsilon_{b2}}{\epsilon_{b1}} \ll 1$ can solve iteratively

to lowest order: $n_a \sin\theta_0 \simeq n_b \sin\theta_2'$

$$\Rightarrow \cos^2\theta_2' = 1 - \sin^2\theta_2' = 1 - \left(\frac{n_a}{n_b}\sin\theta_0\right)^2$$

so to next order

$$n_a \sin\theta_0 \simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2\left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]$$

or $$\sin\theta_2' \simeq \frac{n_a}{n_b}\sin\theta_0 \frac{1}{\left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2\left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]}$$

$$\leq \frac{n_a}{n_b}\sin\theta_0$$

result is that θ_2' is <u>smaller</u> than Snell's law would predict.

(2) for a good conductor, or absorbing region of a dielectric, $\epsilon_{b2} \gg \epsilon_{b1}$

to lowest order

$$n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}} \left[\frac{1}{2} \frac{\epsilon_{b2}}{\epsilon_{b1} \cos\theta_2'} \right]^{1/2} \sin\theta_2'$$

$$n_a \sin\theta_0 = \sqrt{\frac{\mu_b \epsilon_{b2}}{2}} \frac{\sin\theta_2'}{\sqrt{\cos\theta_2'}}$$

very different from Snell's Law!

Snell's law only holds if both media are transparent

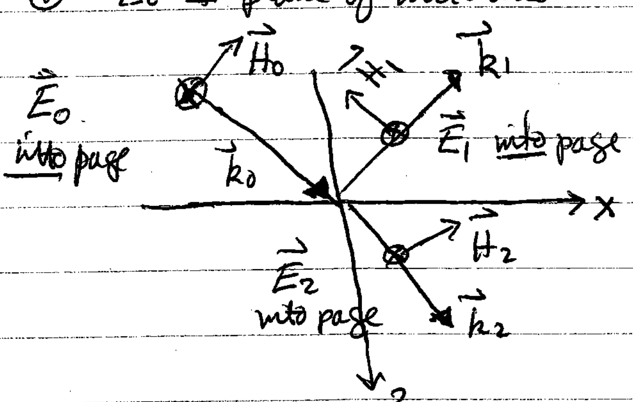
Reflection coefficients

Now we compute the amplitude of the reflected wave to determine how much of incident wave is reflected and how much is transmitted.

Consider two cases ① $\vec{E}_\omega$ is $\perp$ plane of incidence
② $\vec{E}_\omega$ lies in the plane of incidence

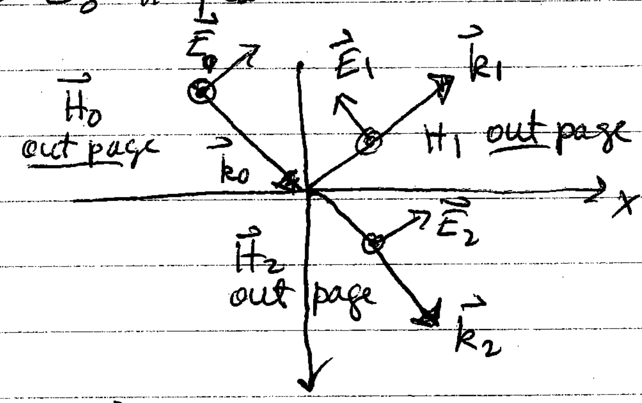
"plane of incidence" is the plane spanned by the wave vector $\vec{k}_0$ and the normal to the interface — in our case it is the xz plane

① $\vec{E}_0 \perp$ plane of incidence



$\Rightarrow \vec{H}_0$ in plane of incidence
all $\vec{E}$'s are in $\hat{y}$ direction

② $\vec{E}_0 \parallel$ plane incidence



$\Rightarrow \vec{H}_0 \perp$ plane of incidence
all the $\vec{H}$'s are in y direction

continuity of y components

1) $E_0 + E_1 = E_2$ | 1) $H_0 + H_1 = H_2$

continuity of x components

$$H_{0x} + H_{1x} = H_{2x}$$

Faraday

$$\frac{i\mu\omega}{c}\vec{H} = i\vec{k}\times\vec{E} \Rightarrow H_x = \frac{k_z c}{\omega\mu}E_y$$

$$E_{0x} + E_{1x} = E_{2x}$$

Ampere

$$-\frac{i\omega\epsilon}{c}\vec{E} = i\vec{k}\times\vec{H} \Rightarrow E_x = \frac{-k_z c}{\omega\epsilon}H_y$$

$\Rightarrow$

2) $\frac{k_{0z}}{\mu_a}(E_0 - E_1) = \frac{k_{2z}}{\mu_b} E_2$

solve (1) and (2) for E_1 and E_2 in terms of E_0

$$E_1 = \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} E_0$$

$$E_2 = \frac{2\mu_b k_{0z}}{\mu_a k_{2z} + \mu_b k_{0z}} E_0$$

2) $\frac{k_{0z}}{\epsilon_a}(H_0 - H_1) = \frac{k_{2z}}{\epsilon_b} H_2$

solve (1) and (2) for H_1 and H_2 in terms of H_0

$$H_1 = \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} H_0$$

$$H_2 = \frac{2\epsilon_b k_{0z}}{\epsilon_a k_{2z} + \epsilon_b k_{0z}} H_0$$

Define reflection coefficient in terms of the transported energy $R = \frac{|E_1|^2}{|E_0|^2} = \frac{|H_1|^2}{|H_0|^2}$

<u>Reflection coefficients</u>

① $\vec{E}_0 \perp$ plane incidence

$$R_\perp = \frac{|E_1|^2}{|E_0|^2} = \left| \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} \right|^2$$

② $\vec{E}_0 \parallel$ plane incidence

$$R_\parallel = \frac{|H_1|^2}{|H_0|^2} = \left| \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} \right|^2$$

Note: above are correct for an <u>arbitrary</u> medium B

i) Consider region of "total reflection"

$\Rightarrow \quad \mathrm{Im}\,\epsilon_b = \epsilon_{b2} \approx 0$

$\mathrm{Re}\,\epsilon_b = \epsilon_{b1} < 0 \quad \Big\} \Rightarrow \vec{k}_2 = i\vec{K}_2$ where $\vec{K}_2$ is real

ie $\vec{k}_2$ pure imaginary

$$\Rightarrow \quad R_\perp = \left| \frac{\mu_b k_{0z} - i\mu_a K_{2z}}{\mu_b k_{0z} + i\mu_a K_{2z}} \right|^2$$

$$R_\parallel = \left| \frac{\epsilon_b k_{0z} - i\epsilon_a K_{2z}}{\epsilon_b k_{0z} + i\epsilon_a K_{2z}} \right|^2$$

both are of the form $\left|\frac{a-ib}{a+ib}\right|^2 = 1$ when a, b real

$\Rightarrow R_\perp = R_\parallel = 1$

Confirms that the material is completely reflecting

ii) Next consider when medium B is transparent

ϵ_b is real and $\epsilon_b > 0$

$$k_{0z} = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a}\cos\theta_0 = \frac{\omega}{c} n_a \cos\theta_0$$

$$k_{2z} = \frac{\omega}{c}\sqrt{\mu_b \epsilon_b}\cos\theta_2 = \frac{\omega}{c} n_b \cos\theta_2$$

Snell's law holds so $n_a \sin\theta_0 = n_b \sin\theta_2$

Can write $R_\perp$ and $R_\parallel$ as functions of θ_0

for simplicity take $\mu_a = \mu_b = 1$

① $R_\perp = \left|\dfrac{n_a\cos\theta_0 - n_b\cos\theta_2}{n_a\cos\theta_0 + n_b\cos\theta_2}\right|^2 = \left(\dfrac{\cos\theta_0 - \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}{\cos\theta_0 - \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}\right)^2$

$$= \left(\frac{\sin\theta_2\cos\theta_0 - \sin\theta_0\cos\theta_2}{\sin\theta_2\cos\theta_0 + \sin\theta_0\cos\theta_2}\right)^2$$

$$R_\perp = \left(\frac{\sin(\theta_0-\theta_2)}{\sin(\theta_0+\theta_2)}\right)^2$$

for $\theta_0 = 0$, ie normal incidence, $\theta_2 = 0$

$\Rightarrow R_\perp = \left(\dfrac{n_a - n_b}{n_a + n_b}\right)^2$ if $n_a = n_b$, no reflection!
(not surprising!)

② $R_\parallel = \left(\dfrac{\epsilon_b n_a\cos\theta_0 - \epsilon_a n_b\cos\theta_2}{\epsilon_b n_a\cos\theta_0 + \epsilon_a n_b\cos\theta_2}\right)^2$ use $\sqrt{\epsilon_b} = n_b$
$\sqrt{\epsilon_a} = n_a$

$$= \left(\frac{n_b\cos\theta_0 - n_a\cos\theta_2}{n_b\cos\theta_0 + n_a\cos\theta_2}\right)^2$$

$$= \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_0\cos\theta_0 - \sin\theta_2\cos\theta_2}{\sin\theta_0\cos\theta_0 + \sin\theta_2\cos\theta_2}\right)^2$$

$R_\parallel = \left(\dfrac{\tan(\theta_0-\theta_2)}{\tan(\theta_0+\theta_2)}\right)^2$ ← after some algebra!

for $\theta_0 = 0$, then $\theta_2 = 0$

$$R_{\parallel} = \left(\frac{\epsilon_b n_a - \epsilon_a n_b}{\epsilon_b n_a + \epsilon_a n_b}\right)^2 = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2 \quad \text{same as } R_{\perp}$$

So for $\theta_0 = 0$, $R_{\parallel} = R_{\perp}$ — this must be so since for $\theta_0 = 0$ there is no distinction between the $\perp$ and $\parallel$ cases

If $n_b = n_a$, $R_{\perp} = R_{\parallel} = 0$ no reflective wave

When $\theta_0 + \theta_2 = \pi/2$, then $\tan(\theta_0 + \theta_2) \to \infty$ and $R_{\parallel} = 0$

This occurs at an angle of incidence known as Brewster's angle θ_B, determined by

$$\underset{\substack{\uparrow \\ \theta_0}}{n_a \sin \theta_B} = n_b \sin\Big(\underset{\substack{\uparrow \\ \theta_2}}{\tfrac{\pi}{2} - \theta_B}\Big) = n_b \cos \theta_B$$

$$\Rightarrow \boxed{\tan \theta_B = \frac{n_b}{n_a}}$$

For incident wave at θ_B, reflected wave always has $\vec{E}_1 \perp$ plane of incidence, since $R_{\parallel} = 0$. If incoming wave has $\vec{E}_0 \parallel$ plane of incidence, then it gets completely transmitted. If $\vec{E}_0$ in general direction, reflected wave is always linearly polarized with $\vec{E}_1 \perp$ plane of incidence. — This is one method to create polarized light wave.

Kramers - Kronig Relation

We saw that $\vec{P}_\omega = \alpha(\omega)\vec{E}_\omega$

Causal response ie $\tilde{\alpha}(t) = 0$ for $t < 0$

$\Rightarrow$ $\alpha(\omega)$ has no poles in upper half of complex ω plane (UHP)

For any complex $\bar{\omega}$ in upper half of complex ω plane,

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i}\oint \frac{\alpha(\omega')}{\omega'-\bar{\omega}}\,d\omega'$$

since no poles of α in UHP

contour along real axis, closed at infinity in UHP. The closing ~~top~~ semicircle at infinity gives no contribution assuming $\alpha(\omega)$ decays quickly enough as $|\omega| \to \infty$

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \,\frac{\alpha(\omega')}{\omega'-\bar{\omega}}$$

Now consider $\bar{\omega} = \omega + i\delta$ where ω and δ are real and $\delta \to 0$

$$\alpha(\omega) = \lim_{\delta \to 0} \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \,\frac{\alpha(\omega')}{\omega'-\omega-i\delta}$$

Now $$\frac{1}{\omega'-\omega-i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + i\pi\,\delta(\omega'-\omega)$$

principle part

$$\Rightarrow \alpha(\omega) = \frac{1}{\pi i} P\int_{-\infty}^{\infty} \frac{\alpha(\omega')\,d\omega'}{\omega'-\omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\,\alpha(\omega) &= \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\alpha(\omega')\,d\omega'}{\omega'-\omega} \\ \mathrm{Im}\,\alpha(\omega) &= \frac{-1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\alpha(\omega')\,d\omega'}{\omega'-\omega} \end{aligned}}$$

Kramer Kronig relations

If know Re α <u>or</u> Im α can reconstruct full complex α

True for any causal response function