Radiation From Moving Charges

In Lorentz gauge $\frac{1}{c}\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

$$\left.\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\frac{4\pi}{c}\vec{J} \end{aligned}\right\} \text{wave equation with source}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

If we can solve wave equation with source (inhomogeneous wave equation) then we are in principle done! To do this we want to find the Green's function for the wave equation

Recall from statics: $\nabla^2\phi = -4\pi\rho$

Greens function satisfies $\nabla^2 G(\vec{r}) = -4\pi\delta(\vec{r})$

then $\phi(\vec{r}) = \int d^3r'\, G(\vec{r}-\vec{r}')\,\rho(\vec{r}') + \phi_0$

solution for infinite volume that vanishes as $r\to\infty$ is

$$G(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} \qquad \nabla^2\phi_0 = 0$$

For wave equation we want solution to

$$\nabla^2 G(\vec{r},t;\vec{r}',t') - \frac{1}{c^2}\frac{\partial^2 G(\vec{r}t;\vec{r}'t')}{\partial t^2} = -4\pi\,\delta(r-r')\,\delta(t-t')$$

Then we will have
$$\left\{\begin{aligned} \phi(\vec{r},t) &= \int d^3r'\, G(\vec{r},t;\vec{r}'t')\,\rho(\vec{r}',t') + \phi_0 \\ \vec{A}(\vec{r},t) &= \frac{1}{c}\int d^3r'\, G(\vec{r}t;\vec{r}'t)\,\vec{J}(\vec{r}',t') + \vec{A}_0 \end{aligned}\right.$$

to treat the poles on the real axis so that $G(\vec{r},t)$ will have the desired behavior.

What we want is for $G(\vec{r},t)$ to be causal, ie $G(\vec{r},t)=0$ for $t<0$, so $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ depend only on the values of the sources at earlier times $t'<t$.

$$\int d^3k\, e^{i\vec{k}\cdot\vec{r}}\,\tilde{G}(\vec{k},\omega) = 2\pi\int_0^{2\pi} d\theta \sin\theta \int_0^\infty dk\, k^2\, e^{ikr\cos\theta}\,\tilde{G}(k,\omega)$$

$$\mu \equiv \cos\theta$$

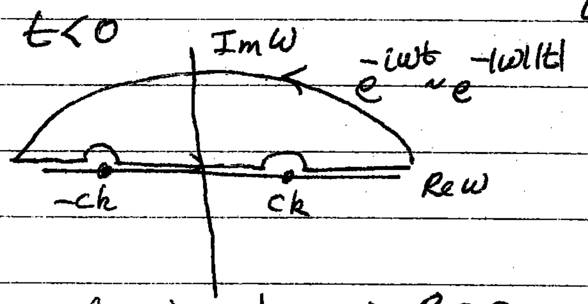
$$= 2\pi\int_{-1}^{1} d\mu \int_0^\infty dk\, k^2\, e^{ikr\mu}\,\tilde{G}(k,\omega)$$

$$= 4\pi\int_0^\infty dk\, k^2\, \frac{\sin kr}{kr}\,\tilde{G}(k,\omega)$$

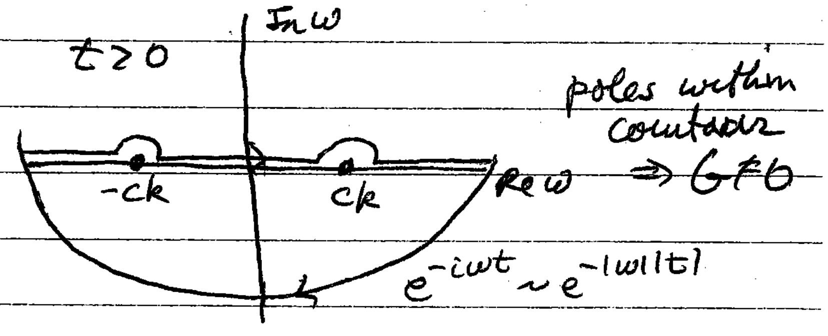
$$G(\vec{r},t) = -\frac{c^2}{\pi^2}\int_0^\infty dk\, k^2\,\frac{\sin kr}{kr}\int_C \frac{e^{-i\omega t}}{(\omega+ck)(\omega-ck)}\,d\omega$$

contour along real axis, but deformed to go around the poles

for $t<0$, $e^{-i\omega t}$ will decay exponentially fast for large $|\omega|$ in the upper half complex (UHP) ω plane $\Rightarrow$ can close contour in UHP for $t<0$. If we want $G=0$ for $t<0$, there should therefore be no poles in UHP. The contour C we want is therefore:



no poles in contour ⇒ $G=0$



poles within contour ⇒ $G\neq 0$

with this convention for the contour C we can evaluate the ω-integral using Cauchy's residue theorem

$$\int \frac{e^{-i\omega t}\, d\omega}{(\omega+ck)(\omega-ck)} = -2\pi i \left[\frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck} \right]$$

$$= -\frac{2\pi \sin(ckt)}{ck}$$

$$G(\vec{r},t) = \frac{2c}{\pi r}\int_0^\infty dk\, \sin(kr)\sin(ckt) = \frac{c}{\pi r}\int_{-\infty}^{\infty} dk\, \frac{(e^{ikr}-e^{-ikr})(e^{ickt}-e^{-ickt})}{(-4)}$$

$$= -\frac{c}{2r}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{ e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k} \right.$$

each integral would give a δ-function, but for 1st two terms $\delta(r+ct)=0$ since here $t>0$ (by definition) and $r=|\vec{r}|\geq 0$ so the arguement will never vanish.

$$\boxed{G(\vec{r},t) = \frac{c}{r}\,\delta(r-ct) = \frac{\delta(t-r/c)}{r}}$$

using $\delta(ax)=\frac{1}{a}\delta(x)$

$$G(\vec{r}t,\vec{r}'t') = \begin{cases} \dfrac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} & t>0 \\ 0 & t<0 \end{cases}$$

Green's function for wave equation in free space

$G \neq 0$ only on "light cone" that ~~connects~~ emanates from (r',t'), ie when $|\vec{r}-\vec{r}'| = c(t-t')$.

Signal from source at (r',t') travels with c

$$\phi(\vec{r},t) = \phi_0(\vec{r},t) + \int d^3r' \int_{-\infty}^{t} dt' \; \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \; \rho(\vec{r}',t')$$

$$A(\vec{r},t) = \vec{A}_0(\vec{r},t) + \int \frac{d^3r'}{c} \int_{-\infty}^{t} dt' \; \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \; \vec{j}(\vec{r}',t')$$

Apply to a <u>single moving point charge</u>

$$\rho(\vec{r},t) = q\,\delta(\vec{r} - \vec{r}_0(t))$$

$$\vec{j}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r}-\vec{r}_0(t)) \qquad \text{where } \vec{v}(t) = \frac{d\vec{r}_0}{dt}$$

Then

$$\phi(\vec{r},t) = q \int dt' \; \frac{\delta\left(t - t' - \frac{1}{c}|\vec{r}-\vec{r}_0(t')|\right)}{|\vec{r}-\vec{r}_0(t')|}$$

because of the $\vec{r}_0(t')$ in the argument of the $\delta()$ function the t' dependence is not of the simple form $t' - t_0$.

We can write

$$g(t') \equiv t' + \frac{1}{c}\,|\vec{r} - \vec{r}_0(t')|$$

then

$$\phi(\vec{r},t) = q \int dt' \; \frac{\delta(t - g(t'))}{|\vec{r}-\vec{r}_0(t')|}$$

$$= q \int \frac{\delta(t - g(t'))}{|\vec{r}-\vec{r}_0(t')|} \, dg \left(\frac{dt'}{dg}\right)$$

$$= \left. \frac{q}{|\vec{r}-\vec{r}_0(t')|} \; \frac{1}{(dg/dt')} \right|_{t' \text{ such that } g(t') = t}$$

$$g(t') = t' + \frac{1}{c}\sqrt{[x-x_0(t')]^2 + [y-y_0(t')]^2 + [z-z_0(t')]^2}$$

$$\frac{dg}{dt'} = 1 + \frac{1}{c\,|r-r_0(t')|}\left\{[x-x_0(t')]\left(-\frac{dx_0}{dt'}\right) + \cdots\right\}$$

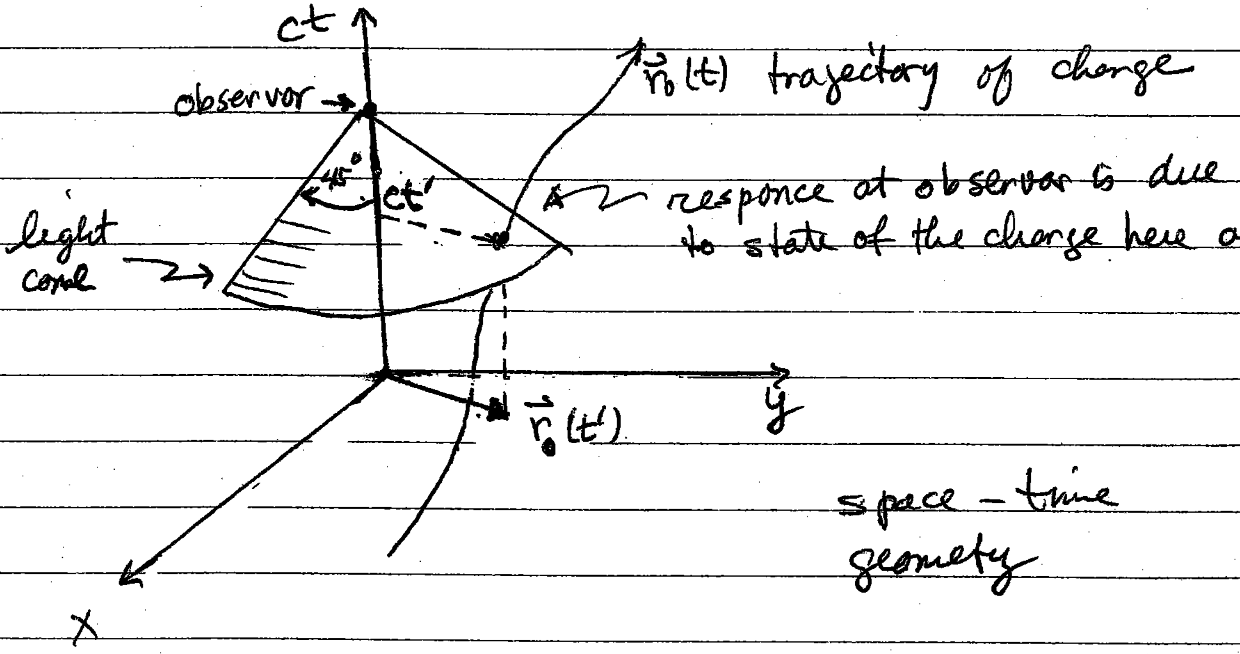
$$= 1 - \frac{1}{c}\,\hat{n}(t')\cdot\vec{v}(t')$$

where $\hat{n}(t') = \dfrac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|}$

$$\left.\begin{aligned} \phi(\vec{r},t) &= \frac{q}{|\vec{r}-\vec{r}_0(t')|\left[1-\frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')\right]} \\ \vec{A}(\vec{r},t) &= \frac{q\,\vec{v}(t')/c}{|\vec{r}-\vec{r}_0(t')|\left[1-\frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')\right]} \end{aligned}\right\}\ \underline{\text{Lienard - Wiechert Potentials}}$$

where t' is determined by the condition

$$t - t' = \frac{1}{c}|\vec{r}-\vec{r}_0(t')|$$

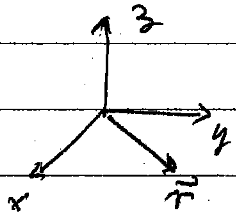


space - time geometry

For charge moving with constant velocity along $\hat{z}$

$\vec{r}_0(t) = vt\,\hat{z}$ $\qquad \vec{v} = \frac{d\vec{r}_0}{dt} = v\hat{z}$

For observer at position $\vec{r}$ (in xy plane), time t, the fields will be determined by the charge at time t' such that



$$t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c} = 0$$

$$t - t' - \frac{\sqrt{r^2 + v^2 t'^2}}{c} = 0$$

$$(t-t')^2 = t^2 + t'^2 - 2tt' = \frac{r^2 + v^2 t'^2}{c^2}$$

$$(1 - v^2/c^2)\,t'^2 - 2tt' + t^2 - \frac{r^2}{c^2} = 0$$

let $\gamma = (1 - v^2/c^2)^{-1/2}$

$$t' - 2\gamma^2 t\,t' + \gamma^2\frac{(c^2t^2 - r^2)}{c^2} = 0$$

$$t' = \gamma^2 t \pm \sqrt{\gamma^4 t^2 - \gamma^2 t^2 + \gamma^2 \frac{r^2}{c^2}}$$

$$= \gamma^2 t \pm \sqrt{\gamma^2\left(\gamma^2 t^2 - t^2 + \frac{r^2}{c^2}\right)}$$

$$\gamma^2 - 1 = \frac{1}{1 - v^2/c^2} - 1 = \frac{v^2/c^2}{1 + v^2/c^2}$$

$$= \gamma^2 t \pm \gamma\sqrt{t^2\gamma^2\left(\frac{v^2}{c^2}\right) + \frac{r^2}{c^2}}$$

$$t' = \gamma^2 t \pm \frac{\gamma^2}{c}\sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

consider $t = 0$. solution should give $t' < 0$

$\Rightarrow$ $(-)$ sign is the solution we want

$$t' = \gamma^2 t - \frac{\gamma^2}{c}\sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

$$\phi(\vec r, t) = \frac{q}{|\vec r - \vec r_0(t')| \left[1 - \frac{1}{c}\hat{n}(t') \cdot \vec v\right]}$$

[illegible]

$$|\vec r - \vec r_0(t')| = r^2 + v^2 t'^2 = c(t - t')$$ from condition that determines t'

$$(\vec r - \vec r_0(t')) \cdot \vec v = -\vec r_0(t') \cdot \vec v$$ for $\vec v = v\hat z$

$$= -v^2 t'$$ $\vec r$ in xy plane

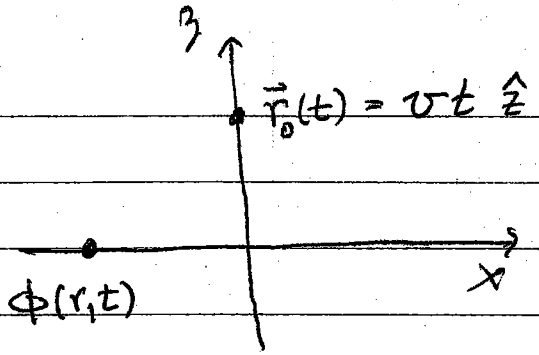
$$\phi(\vec r, t) = \frac{q}{c(t - t')\left[1 + \frac{v^2 t'}{c^2(t - t')}\right]}$$

$$= \frac{q}{c(t - t') + \frac{v^2 t'}{c}} = \frac{q}{c\left[t - \left(1 - \frac{v^2}{c^2}\right)t'\right]}$$

$$= \frac{q}{c\left(t - \frac{t'}{\gamma^2}\right)} = \frac{q}{c\,\frac{1}{c}\sqrt{v^2 t^2 + r^2/\gamma^2}}$$

$$\left.\begin{aligned} \phi(\vec r, t) &= \frac{q}{\sqrt{v^2 t^2 + r^2/\gamma^2}} \\ \vec A(\vec r, t) &= \frac{q\vec v}{c\sqrt{v^2 t^2 + r^2/\gamma^2}} \end{aligned}\right]$$

solutions for
$\vec r$ in xy plane
when charge passes
through xy plane
at $t = 0$



at x
~~potential from q at $vt\hat{z}$~~

potential at pt $\vec{r}$ in xy plane at time t, when charge is at $\vec{r}_0 = vt\,\hat{z}$, looks almost like static Coulomb potential, which would be $\dfrac{q}{\sqrt{r^2 + v^2t^2}}$

But instead, it is

$$\frac{q}{\sqrt{v^2t^2 + \left(\frac{r}{\gamma}\right)^2}}$$

looks like the transverse direction has contracted by a factor γ!

Such considerations led Lorentz to discover the Lorentz transformation, <u>before</u> Einstein proposed his theory of special relativity