For arbitrary charge distributions - not pure harmonic

For $\vec{p}_\omega e^{-i\omega t}$ pure harmonic oscillation, we found the radiated fields in electric dipole approx are

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}, \quad \vec{B} = \vec{B}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega = -k^2 \frac{e^{ikr}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega) = -\frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$\vec{B}_\omega = k^2 \frac{e^{ikr}}{r} (\hat{r} \times \vec{p}_\omega) = \frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} (\hat{r} \times \vec{p}_\omega)$$

as $k = \frac{\omega}{c}$

For an arbitrarily time varying charge distribution with electric dipole moment

$$\vec{p}(t) = \int \frac{d\omega}{2\pi} \vec{p}_\omega e^{-i\omega t}$$

then solution for fields given by superposition

$$\vec{E}(\vec{r},t) = \int \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$= -\int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2}\right) \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$= \frac{-1}{c^2 r} \hat{r} \times \left[\hat{r} \times \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega \, \omega^2\right]$$

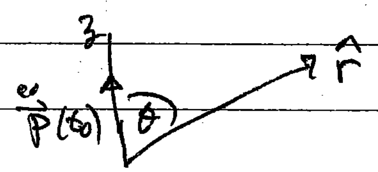
$$= \frac{1}{c^2 r} \hat{r} \times \left[\hat{r} \times \frac{\partial^2}{\partial t^2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega\right]$$

$$\boxed{\vec{E}(\vec{r},t) = \frac{1}{c^2 r}\,\hat{r}\times\left[\hat{r}\times\ddot{\vec{p}}(t-r/c)\right]} \qquad \ddot{\vec{p}} = \frac{d^2\vec{p}}{dt^2}$$

define $t_0 \equiv t - r/c$ = "retarded time"

in spherical coords, if $\ddot{\vec{p}}(t_0)$ *is along* $\hat{z}$

$$\vec{E}(\vec{r},t) = \frac{\ddot{p}(t_0)}{c^2 r}\sin\theta\;\hat{\theta}$$



Similarly

$$\vec{B}(\vec{r},t) = \int\frac{d\omega}{2\pi}\,\vec{B}_\omega\, e^{-i\omega t}$$

$$= \int\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-r/c)}}{r}\left(\frac{\omega^2}{c^2}\right)\left(\hat{r}\times\vec{p}_\omega\right)$$

$$= \frac{-1}{c^2 r}\,\hat{r}\times\frac{\partial^2}{\partial t^2}\int\frac{d\omega}{2\pi}\,e^{-i\omega(t-r/c)}\,\vec{p}_\omega$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{-1}{c^2 r}\,\hat{r}\times\ddot{\vec{p}}(t_0)}$$

$$\vec{B}(\vec{r},t) = \frac{\ddot{p}(t_0)}{c^2 r}\sin\theta\;\hat{\varphi} \qquad \text{in spherical coords}$$

<u>Poynting vector</u>

$$\vec{S} = \frac{c}{4\pi}\,\vec{E}\times\vec{B} = \frac{c}{4\pi}\left(\frac{1}{c^2 r}\right)^2\left[\ddot{p}(t_0)\right]^2\sin^2\theta\;\hat{r}$$

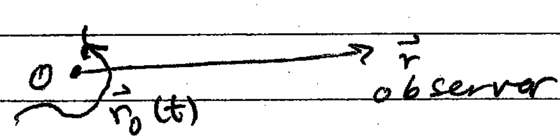
Total power radiated through a sphere of radius r is

$$\mathcal{P} = \oint da\, \hat{r}\cdot\vec{S} = 2\pi \int_0^{\pi} d\theta \sin\theta\, r^2\, \hat{r}\cdot\vec{S}$$

$$= \frac{[\ddot{p}(t_0)]^2}{2c^3} \underbrace{\int_0^{\pi} d\theta \sin^3\theta}_{4/3}$$

$$\boxed{\mathcal{P} = \frac{2\,[\ddot{p}(t_0)]^2}{3c^2}}$$

For a point charge moving along a trajectory $\vec{r}_0(t)$



$$\vec{p}(t) = q\vec{r}_0(t)$$
$$\ddot{\vec{p}}(t) = q\ddot{\vec{r}}_0(t) = q\vec{a}(t)$$
↑ acceleration

$$\boxed{\mathcal{P} = \frac{2}{3}\frac{q^2 a^2(t_0)}{c^3}}$$

Larmor's formular

← total power passing through a sphere of radius r at time t is due to acceleration at retarded time $t_0 = t - r/c$

power radiated ~ (acceleration)^2

Larmor's formular above only holds in the non-relativistic limit since it is based on the electric dipole approx.

To go beyond non-relativistic limit, one can do the following:

1) transform to a new inertial frame of reference in which the charge is <u>instantaneously</u> at rest
2) apply non-relativistic Larmor formula in this frame
3) transform back to "lab" frame

<u>Back to non-relativistic case</u>

radiation fields from moving point charge

$$\vec{E}(\vec{r},t) = \frac{q}{c^2 r}\,\hat{r} \times (\hat{r} \times \vec{a}(t_0))$$

$$\vec{B}(\vec{r},t) = -\frac{q}{c^2 r}\,(\hat{r} \times \vec{a}(t_0))$$

<u>Poynting vector</u>

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = -\frac{c}{4\pi}\left(\frac{q}{c^2 r}\right)^2 \left[\hat{r}\times(\hat{r}\times\vec{a})\right]\times\left[\hat{r}\times\vec{a}\right]$$

$$= \frac{c}{4\pi}\left(\frac{q}{c^2 r}\right)^2 \left\{ \hat{r}\,(\hat{r}\times\vec{a})^2 - (\hat{r}\times\vec{a})\left[\hat{r}\cdot(\hat{r}\times\vec{a})\right]\right\}$$

$$= \frac{q^2}{4\pi c^3}\left[\frac{\hat{r}\times\vec{a}(t_0)}{r^2}\right]^2 \hat{r}$$

$$\boxed{\vec{S} = \frac{q^2}{4\pi c^3 r^2}\left(a^2 - (\hat{r}\cdot\vec{a})^2\right)\hat{r} = \frac{q^2}{4\pi c^3}\frac{a^2(t_0)}{r^2}\sin^2\theta\,\hat{r}}$$

$\vec{a}$ evaluated at t_0

for $\vec{a} = a\hat{z}$