Electromagnetic Potentials & Gauge Invariance

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Consider first <u>statics</u>.

<u>Electrostatics</u>

$$\vec{\nabla}\times\vec{E} = 0 \qquad \text{since } \frac{\partial\vec{B}}{\partial t} = 0 \text{ for statics}$$

From vector calculus we know that if the curl of a vector is ~~else~~ everywhere zero, then we can alway write that vector field as the gradient of some scalar function ϕ

$$\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{\nabla}\times E = -\vec{\nabla}\times(\vec{\nabla}\phi) = 0$$

ϕ is <u>electrostatic potential</u>

Gauss Law becomes

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot(\vec{\nabla}\phi) = -\nabla^2\phi = 4\pi\rho$$

$$\boxed{\nabla^2\phi = -4\pi\rho} \quad \text{Poisson's Equation}$$

In regions where $\rho=0$, we have

$$\nabla^2\phi = 0 \qquad \text{Laplace's Equation}$$

In our discussion of Coulomb's Law we saw that the electric field from a distribution of localized charges was

$$\vec{E}(\vec{r}) = \int d^3r' \, \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$= -\vec{\nabla}\left[\int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] = -\vec{\nabla}\phi$$

We therefore see that the solution to Poisson's equ for a localized charge distribution ρ (with $\vec{E}=0$ as $\vec{r}\to\infty$) is

$$\phi(\vec{r}) = \int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

We will soon spend a fair amount of time learning new ways to solve $\nabla^2\phi=\rho$, both for arbitrary ρ where we want an approx to the above integral (multipole expansion), and for cases where ϕ or $\vec{\nabla}\phi$ are predetermined on the surfaces of specified regions of space, such as conducting surfaces (boundary value problems).

Magnetostatics

$$\vec{\nabla}\cdot\vec{B} = 0$$

From vector calculus we know that if the divergence of a vector field vanishes everywhere, then it can always be written as the curl of another vector field $\vec{A}$

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

$\vec{A}$ is the magnetic vector potential

This remains true in general - not just in magnetostatics

Ampere's Law becomes

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} \qquad \left(\text{in magnetostatics } \frac{\partial\vec{E}}{\partial t} = 0\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \frac{4\pi\vec{j}}{c}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}$$

magnetostatic gauge invarience

There are many possible vector potentials $\vec{A}$ that result in the same $\vec{B}$. If $\vec{A}$ is such that $\vec{\nabla}\times\vec{A} = \vec{B}$, then $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ also gives $\vec{\nabla}\times\vec{A}' = \vec{B}$, since $\vec{\nabla}\times\vec{\nabla}\chi = 0$ for any scalar function $\chi(\vec{r})$.

Therefore we can always choose to represent $\vec{B}$ by a vector potential $\vec{A}$ such that $\vec{\nabla}\cdot\vec{A} = 0$.

proof: Suppose we had $\vec{B} = \vec{\nabla}\times\vec{A}$ for some $\vec{A}$ with $\vec{\nabla}\cdot\vec{A} = D(\vec{r}) \neq 0$. Construct an $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ with χ chosen as follows:

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\chi = 0 \Rightarrow \nabla^2\chi = -\vec{\nabla}\cdot\vec{A} = D$$

Solve for χ, for example

$$\chi(\vec{r}) = \int d^3r' \frac{D(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}$$

we thus have constructed an $\vec{A}'$ such that $\vec{\nabla}\times\vec{A}' = \vec{B}$ and $\vec{\nabla}\cdot\vec{A}' = 0$.

This freedom to choose various $\vec{A}$'s that give the same $\vec{B}$ is called gauge invariance. Imposing a particular additional constraint on $\vec{A}$ that removes this freedom is called fixing the gauge. The choice $\vec{\nabla}\cdot\vec{A} = 0$ is usually known as the Coulomb gauge (or sometimes the Landau gauge). Going from $\vec{A}$ to $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ is called making a gauge transformation.

"Working in the Coulomb gauge" with $\vec{\nabla}\cdot\vec{A} = 0$, Amperes Law becomes

$$\boxed{\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{j}}$$ Poisson's Equ.

For a localized current density

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Back to dynamics

$$\vec{\nabla}\cdot\vec{B}=0 \quad\Rightarrow\quad \boxed{\vec{B}=\vec{\nabla}\times\vec{A}} \quad \text{remains true}$$

But now instead of $\vec{\nabla}\times\vec{E}=0$ we have

$$\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$$

$$\Rightarrow\quad \vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})=0$$

$$\Rightarrow\quad \vec{\nabla}\times\left(\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)=0$$

$\Rightarrow$ there exists a scalar potential ϕ such that

$$\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\phi \quad \text{or} \quad \boxed{\vec{E}=-\vec{\nabla}\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$$

Gauss's law for electric field now becomes

$$\vec{\nabla}\cdot\vec{E}=4\pi\rho = -\nabla^2\phi-\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=4\pi\rho$$

$$\boxed{\nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=-4\pi\rho}$$ Gauss law in terms of electromagnetic potentials

Ampere's law becomes

$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{j}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\frac{4\pi}{c}\vec{j}-\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\phi-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\boxed{-\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)}$$

or $$\boxed{-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}$$

<u>gauge invarience</u>

As before, we can always construct $\vec{A}' = \vec{A} + \vec{\nabla}\chi$, for any scalar function χ, that gives the same $\vec{B}$. But since $\vec{A}$ now also enters expression for $\vec{E}$, we need to make sure that if we change $\vec{A}$ to $\vec{A}'$, we must make some corresponding change ϕ to ϕ' so that $\vec{E}$ does not change.

$$\left[\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\chi \\ \phi' &= \phi - \frac{1}{c}\frac{\partial\chi}{\partial t} \end{aligned}\right] \quad \text{gauge transformation}$$

For any scalar χ, the above $\vec{A}'$ and ϕ' give the same values of $\vec{E}$ and $\vec{B}$ as $\vec{A}$ and ϕ.

<u>Proof:</u> $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\chi = \vec{\nabla}\times\vec{A} = \vec{B}$

$$\left(-\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial\vec{A}'}{\partial t}\right) = -\vec{\nabla}\phi + \frac{1}{c}\vec{\nabla}\frac{\partial\chi}{\partial t} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\chi$$

$$= \left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = \vec{E}$$

As before, we can <u>fix the gauge</u> by imposing some additional constraint on $\vec{A}$ and ϕ. There are <u>two</u> popular choices:

1) Lorentz Gauge

gauge constraint: require $\frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

Then Gauss' Law becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -4\pi\rho$$

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho}$$

Ampere's Law becomes

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}_{=0}$$

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}}$$

The combination $-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \equiv \Box^2$
is the wave equation operator.

In Lorentz gauge, $\vec{A}$ and ϕ satisfy the inhomogeneous wave equations:

$$\boxed{\begin{aligned}\Box^2\vec{A} &= \frac{4\pi}{c}\vec{J}\\ \Box^2\phi &= 4\pi\rho\end{aligned}}$$

when $\vec{J}=0$, $\rho=0$
electromagnetic waves are solution!

Note: Lorentz gauge condition does not uniquely determine $\vec{A}$ and ϕ. If one ~~constructs~~ has $\vec{A}$ and ϕ obeying Lorentz gauge condition, and then constructs

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$
$$\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$$

Then $\vec{A}'$ and ϕ' will also be in Lorentz gauge provided $\Box^2 \chi = 0$ (proof left to reader)

2) Coulomb Gauge

gauge constraint: require $\vec{\nabla}\cdot\vec{A} = 0$
if $\vec{A}$ is in the Coulomb Gauge, then $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ will also be in Coulomb gauge provided $\nabla^2\chi = 0$.

Then Gauss' law becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \boxed{\nabla^2\phi = -4\pi\rho} \quad \text{same as electrostatics!}$$

$$\Rightarrow \phi(\vec{r},t) = \int d^3r' \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

no matter what motion the source $\rho(\vec{r},t)$ has! ϕ is given by the instantaneous Coulomb potential even though electromagnetic fields have a finite velocity of propagation c!

Ampere's Law becomes:

$$-\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right)$$

$$\Rightarrow \Box^2 A = \frac{4\pi}{c}\vec{J} - \frac{1}{c}\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right)$$

where $\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right) = \vec{\nabla}\left[\int d^3r' \frac{\partial \rho}{\partial t}\frac{1}{|\vec{r}-\vec{r}'|}\right]$

$$= -\vec{\nabla}\left[\int d^3r' \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\right]$$ by continuity eqn.

To see the meaning of this term, recall - any vector function $\vec{J}$ can be written as the sum of a curlfree and a divergenceless part

$\vec{J} = \vec{J}_{\parallel} + \vec{J}_{\perp}$ where $\vec{\nabla}\times\vec{J}_{\parallel} = 0$ curlfree

$\vec{\nabla}\cdot\vec{J}_{\perp} = 0$ divergenceless

where

$$\vec{J}_{\parallel}(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla}\int d^3r' \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ longitudinal part

$$\vec{J}_{\perp}(\vec{r}) = \frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ transverse part

Proof: Suppose one has a $\vec{f}(\vec{r})$ with

$$\vec{\nabla}\cdot\vec{f} = 4\pi D(\vec{r})$$
$$\vec{\nabla}\times\vec{f} = 4\pi \vec{C}(\vec{r})$$

write $\vec{f} = \vec{f}_{\parallel} + \vec{f}_{\perp}$ where $\vec{\nabla}\times\vec{f}_{\parallel} = 0$ and $\vec{\nabla}\cdot\vec{f}_{\perp} = 0$

$$\Rightarrow \vec{\nabla}\cdot\vec{f} = \vec{\nabla}\cdot\vec{f}_{\parallel} = 4\pi D(\vec{r})$$

Since $\vec{\nabla}\times\vec{f}_{\parallel} = 0$ we know the unique solution for $\vec{f}_{\parallel}$ is

$$\vec{f}_{\parallel} = -\vec{\nabla}\int d^3r' \frac{D(r')}{|\vec{r}-\vec{r}'|} = -\frac{1}{4\pi}\vec{\nabla}\int d^3r' \frac{(\vec{\nabla}'\cdot\vec{f}(\vec{r}'))}{|\vec{r}-\vec{r}'|}$$

[solution to $\left\{\begin{array}{l}\vec{\nabla}\cdot\vec{f}_{\parallel} = 4\pi D \\ \vec{\nabla}\times\vec{f}_{\parallel} = 0\end{array}\right\}$ is just like Coulomb solution for electrostatics]

Next:

$$\vec{\nabla}\times\vec{f} = \vec{\nabla}\times\vec{f}_{\perp} = 4\pi\vec{C}(\vec{r})$$

$\vec{\nabla}\cdot\vec{f}_{\perp} = 0 \Rightarrow \vec{f}_{\perp} = \vec{\nabla}\times\vec{W}$ some vector potential $\vec{W}$

choose a "gauge" $\vec{\nabla}\cdot\vec{W} = 0$

then
$$\vec{\nabla}\times\vec{f}_{\perp} = \vec{\nabla}\times(\vec{\nabla}\times\vec{W})$$
$$= -\nabla^2\vec{W} + \vec{\nabla}(\vec{\nabla}\cdot\vec{W})$$
$$= -\nabla^2\vec{W} = 4\pi\vec{C}(\vec{r})$$

$$\Rightarrow \vec{W} = \int d^3r' \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{f}_{\perp}(\vec{r}) = \vec{\nabla}\times\vec{W} = \vec{\nabla}\times\int d^3r' \frac{\vec{C}(\vec{r}')}{|r-r'|}$$

$$= \frac{\vec{\nabla}\times}{4\pi}\int d^3r' \frac{(\vec{\nabla}'\times\vec{f}(r'))}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\vec{\nabla}\times}{4\pi}\int d^3r'\,(+)\,\vec{J}(\vec{r}')\times\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) \quad \text{integrate by parts}$$

$$= \frac{\vec{\nabla}\times}{4\pi}\int d^3r'\,\vec{J}(\vec{r}')\times\vec{\nabla}\left(\frac{-1}{|\vec{r}-\vec{r}'|}\right)$$

$$\vec{J}_\perp(\vec{r}) = \frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int d^3r'\,\frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|}$$

The names "longitudinal" and "transverse" parts are best understood by considering the Fourier transform of $\vec{J}$

$$\vec{J}(\vec{r},t) = \int\frac{d^3k\,d\omega}{(2\pi)^4}\,\vec{J}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Since $\vec{\nabla}\cdot\vec{J}_\perp = 0 \Rightarrow \vec{k}\cdot\vec{J}_\perp(\vec{k},\omega) = 0$ or $\vec{J}_\perp \perp \vec{k}$

$\vec{\nabla}\times\vec{J}_\parallel = 0 \Rightarrow \vec{k}\times\vec{J}_\parallel(\vec{k},\omega) = 0$ or $\vec{J}_\parallel \parallel \vec{k}$

For $\vec{J}(\vec{k},\omega) = \vec{J}_\parallel(\vec{k},\omega) + \vec{J}_\perp(\vec{k},\omega)$

$\vec{J}_\parallel(\vec{k},\omega)$ is just the projection of $\vec{J}(\vec{k},\omega)$ onto $\vec{k}$
$\vec{J}_\perp(\vec{k},\omega) = \vec{J}(k,\omega) - \vec{J}_\parallel(\vec{k},\omega)$ is projection of $\vec{J}(k,\omega)$ onto plane $\perp$ to $\vec{k}$.

Returning to Ampere's law we see that the term

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right) = -\vec{\nabla} \int d^3r' \left[\frac{\vec{\nabla}' \cdot \vec{j}(r',t)}{|\vec{r}-\vec{r}'|}\right]$$

$$= 4\pi \vec{j}_{\parallel}(\vec{r},t)$$

So Amperes law becomes

$$\Box^2 \vec{A} = \frac{4\pi}{c}\vec{j} - \frac{4\pi}{c}\vec{j}_{\parallel}$$

$$\boxed{\Box^2 \vec{A} = \frac{4\pi}{c}\vec{j}_{\perp}}$$

In Coulomb gauge, only the transverse part of $\vec{j}$ serves as a source for $\vec{A}$.

$\vec{A}$ describes the transverse modes, ie the EM radiation (recall in EM waves, the fields are always $\perp$ direction of propagation)

ϕ describes the longitudinal modes

Coulomb gauge is not Lorentz invariant - if $\vec{\nabla}\cdot\vec{A} = 0$ in one inertial reference frame, in general $\vec{\nabla}\cdot\vec{A} \neq 0$ in another.

In Coulomb gauge, if $\rho = 0$, then $\phi = 0$ and

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$