Electrostatic

$-\nabla^2\phi = 4\pi\rho$ with $\vec{E} = -\vec{\nabla}\phi$ (statics only)

physical meaning of the potential ϕ

work done to move a test charge δq from $\vec{r}_1$ to $\vec{r}_2$ in presence of an electric field $\vec{E}$ is

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{F}$$

where $\vec{F}$ is the force required to move the charge. Since $\vec{E}$ exerts a force $\delta q\,\vec{E}$ on the charge, $\vec{F}$ must counterbalance this electric force so we can move the charge quasi statically $\Rightarrow \vec{F} = -\delta q\,\vec{E}$

$$W_{12} = -\delta q\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{E} = \delta q\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{\nabla}\phi = \delta q\left[\phi(\vec{r}_2)-\phi(\vec{r}_1)\right]$$

$$\phi(\vec{r}_2)-\phi(\vec{r}_1) = \frac{W_{12}}{\delta q}$$

difference in potential between two points is the work per unit charge to move a test charge between the two points

Green's Functions - part I

$$-\nabla^2\phi = 4\pi\rho$$

We already know that for a point charge q at position $\vec{r}'$, ie $\rho(\vec{r}) = q\,\delta(\vec{r}-\vec{r}')$, the solution to the above is

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}'|} \qquad \text{ie} \quad -\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = 4\pi\delta(\vec{r}-\vec{r}')$$

We call the special solution for a point source the Green function for the differential operator

$$-\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$$

$G(\vec{r},\vec{r}')$ gives the potential at position $\vec{r}$ due to a unit source at position $\vec{r}'$

Generally, one also has to specify a desired boundary condition for the Green function on the boundary of the system.

For the Coulomb solution for a point charge the implicit boundary condition is that the potential vanish infinitely far from the charge

$$G(\vec{r},\vec{r}') \to 0 \quad \text{as} \quad |\vec{r}-\vec{r}'| \to \infty$$

boundary of the system is taken to infinity

If one knows the Green's function, then one can find the solution for any distribution of sources $\rho(\vec{r})$

$$\phi(\vec{r}) = \int d^3r' \, G(\vec{r},\vec{r}\,')\, \rho(\vec{r}\,')$$

<u>proof:</u> $-\nabla^2\phi = \int d^3r' \left[-\nabla^2 G(\vec{r},\vec{r}\,')\right] \rho(\vec{r}\,')$

$$= \int d^3r' \left[4\pi\,\delta(\vec{r}-\vec{r}\,')\right] \rho(\vec{r}\,')$$

$$= 4\pi\,\rho(\vec{r})$$

We will return to concept of Greens function when we discuss solution of Poisson's eqn in

We will also see Greens functions again when we discuss solution of the inhomogeneous wave equation.

The Coulomb problem as a boundary value problem

Consider a conducting sphere of radius R with net charge q (as $R \to 0$ we get a point charge).
What is $\phi(\vec{r})$? What is $E(\vec{r})$?

Review: Properties of conductors in electrostatics

1) $\vec{E} = 0$ inside conductor - if $\vec{E} \neq 0$ then a current $\vec{j} = \sigma \vec{E}$ flows and it is not statics (σ is conductivity)
2) $\rho = 0$ inside conductor - if $\vec{E} = 0$ inside, then $\vec{\nabla} \cdot \vec{E} = 4\pi\rho = 0$
3) Any net charge on the conductor must lie on the surface - follows from (2)
4) $\phi =$ constant throughout conductor - if $\vec{E} = 0$ then $\vec{E} = -\vec{\nabla}\phi \Rightarrow \phi$ is constant
5) Just outside the conductor, $\vec{E}$ is $\perp$ to surface.
   - If $\vec{E}$ has a component $\parallel$ to surface then it exerts a force on electrons at the surface leading to a surface current - so would not be static

For conducting sphere, $\rho = 0$ for $r > R$ and $r < R$
all charge is on the surface $\Rightarrow \nabla^2\phi = 0$ for $\begin{cases} r > R \\ r < R \end{cases}$

spherical symmetry $\Rightarrow$ expect spherically symmetric solution

$\Rightarrow \phi(\vec{r})$ depends only on $r = |\vec{r}|$

⇒ Solve Laplace's eqn by writing ∇^2 in spherical coords. Only the radial terms do not vanish.

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0$$

$$r^2\frac{d\phi}{dr} = -C_0 \quad \text{a constant}$$

$$\frac{d\phi}{dr} = -\frac{C_0}{r^2}$$

$$\phi(r) = \frac{C_0}{r} + C_1 \quad , \quad C_1 \text{ a constant}$$

"outside" $\quad r > R \quad \phi^{out}(r) = \frac{C_0^{out}}{r} + C_1^{out}$

"inside" $\quad r < R \quad \phi^{in}(r) = \frac{C_0^{in}}{r} + C_1^{in}$

solution "outside" does not necessarily smoothly into the solution "inside" because of the charge layer at $r = R$ that seperates the two regions. We need to determine the constants C_0^{in}, C_0^{out}, C_1^{in}, C_1^{out} by applying <u>boundary conditions</u> corresponding to the physical situation.

① For $r > R$, assume $\phi \to 0$ as $r \to \infty$ – boundary condition at infinity

$$\Rightarrow C_1^{out} = 0$$

$\phi^{out}(r) = \frac{C_0^{out}}{r}$ recover the expected Coulomb form.

2) For $r < R$.

i) We could use the fact that the region $r<R$ is a conductor with $\phi = \text{constant}$ to conclude $C_0^{in} = 0$

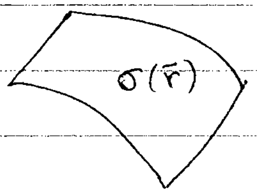
ii) or, if we were dealing with a charged shell instead of a conductor, we could argue as follows:

no charge at origin $r=0$ $\Rightarrow$ expect ϕ should be finite at origin $\Rightarrow$ $C_0^{in} = 0$

So $\quad \phi^{in}(r) = C^{in} \quad$ a constant

3) Now we need boundary condition at $r=R$ where "inside" and "outside" meet.

Review: Electric field and potential at a surface charge layer

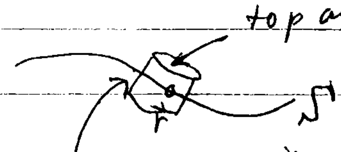


← a general surface S with surface charge density $\sigma(\vec{r})$ for $\vec{r}$ on S. $\sigma(\vec{r})da$ is total charge in area da on surface

i) Take "Gaussian pillbox" surface about point $\vec{r}$ on the surface S

top and bottom areas of pill box da

side view



side of pillbox dl

Gauss' Law in integral form $\oint_S da\, \hat{n}\cdot\vec{E} = 4\pi Q_{\text{enclosed}}$

expect $\vec{E}$ is finite $\Rightarrow$ contribution from sides of pillbox vanish as $dl \to 0$.

$$\oint_S da\, \hat{n}\cdot\vec{E} = \int_{top} da\, \hat{n}\cdot\vec{E} + \int_{bottom} da\, \hat{n}\cdot\vec{E}$$

$$= \left(\hat{n}^{top}\cdot\vec{E}^{top} + \hat{n}^{bottom}\cdot\vec{E}^{bottom}\right)da$$ since da is small

$\vec{E}^{top}$ is electric field at $\vec{r}$ just above the surface S
$\vec{E}^{bottom}$ is electric field at $\vec{r}$ just below the surface S

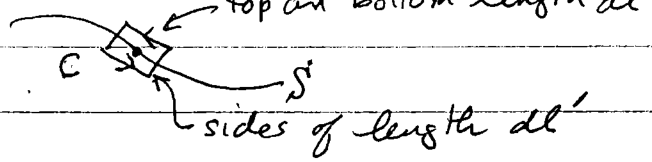
$\hat{n}^{top} \equiv \hat{n}$ is outward normal on top
$\hat{n}^{bottom} = -\hat{n}$ is outward normal on bottom

$$\Rightarrow \left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot\hat{n}\, da = 4\pi Q_{enclosed} = 4\pi\sigma(\vec{r})\, da$$

$$\boxed{\left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot\hat{n} = 4\pi\sigma(\vec{r})}$$ discontinuity in normal component of $\vec{E}$

ii) Take "Amperian loop" C at surface about point $\vec{r}$.

side view



$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \oint_C d\vec{l}\cdot\vec{E}$ since $\vec{E}$ is finite at surface, if take sides $dl' \to 0$ their contribution to integral vanishes

$$\Rightarrow \oint_C d\vec{l}\cdot\vec{E} = \boxed{\left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot d\vec{l} = 0}$$

where $d\vec{l}$ is any infinitesimal tangent to the surface at $\vec{r}$.

$\Rightarrow$ tangential component of $\vec{E}$ is continuous

combine above to write $\boxed{\vec{E}^{top} - \vec{E}^{bottom} = 4\pi\sigma(\vec{r})\,\hat{n}}$

iii) $\vec{E} = -\vec{\nabla}\phi \Rightarrow \phi(r_2) - \phi(r_1) = -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{E}$

Take $\left.\begin{array}{l}\vec{r}_2 \text{ just above } \vec{r} \text{ on surface} \\ \vec{r}_1 \text{ just below } \vec{r} \text{ on surface}\end{array}\right\} d\vec{l} \to 0$

Since $\vec{E}$ is finite $\Rightarrow \int d\vec{l}\cdot\vec{E} \to 0$

$\Rightarrow \boxed{\phi^{top} = \phi^{bottom}}$

potential ϕ is continuous at surface charge layer

can rewrite (i) as

$$\left(-\vec{\nabla}\phi^{top} + \vec{\nabla}\phi^{bottom}\right)\cdot\hat{n} = 4\pi\sigma$$

$$\boxed{-\frac{\partial\phi^{top}}{\partial n} + \frac{\partial\phi^{bottom}}{\partial n} = 4\pi\sigma}$$

directional derivative of ϕ in direction $\hat{n}$

discontinuity in normal derivative of ϕ at surface

Apply to conducting sphere

ϕ continuous $\Rightarrow \phi^{in}(R) = \phi^{out}(R)$

$$C_1^{in} = \frac{C_0^{out}}{R}$$

only one unknown left

normal derivative of ϕ is discontinuous

$$-\frac{\partial \phi^{top}}{\partial n} + \frac{\partial \phi^{bottom}}{\partial n} = 4\pi\sigma$$

here $\hat{n} = \hat{r}$ the radial direction

$$\left[-\frac{d\phi^{out}}{dr} + \frac{d\phi^{in}}{dr} \right]_{r=R} = 4\pi\sigma$$

but $\frac{d\phi^{in}}{dr} = 0$ as ϕ^{in} = constant

$$-\left.\frac{d\phi^{out}}{dr}\right|_{r=R} = 4\pi\sigma$$

charge q is uniformly distributed on surface at R

$$-\frac{d}{dr}\left(\frac{C_0^{out}}{r}\right)_{r=R} = \frac{C_0^{out}}{R^2} = 4\pi\sigma = 4\pi\left(\frac{q}{4\pi R^2}\right) = \frac{q}{R^2}$$

$$\Rightarrow C_0^{out} = q \quad , \quad C^{in} = \frac{C^{out}}{R} = \frac{q}{R}$$

$$\phi(r) = \begin{cases} \frac{q}{R} & r < R \quad \text{inside} \\ \frac{q}{r} & r > R \quad \text{outside} \end{cases}$$

$$\Rightarrow \quad \vec{E} = -\vec{\nabla}\phi = -\frac{d\phi}{dr} = \begin{cases} 0 & r<R \quad \text{inside} \\ \frac{q}{r^2} & r>R \quad \text{outside} \end{cases}$$

we get familiar Coulomb solution!

Summary We can view the preceeding solution for ϕ^{out} as solving Laplace's eqn $\nabla^2\phi = 0$ subject to a specified boundary condition on the normal derivative of ϕ at the boundary $r = R$ of the "outside" region of the system.

Alternate problem:

Another physical situation would be to connect conducting sphere to a battery that charges the sphere to a fixed voltage ϕ_0 (stat volts!) with respect to ground $\phi = 0$ at $r \to \infty$.

As before, outside the sphere $\phi = \frac{C_0}{r}$
Now the boundary condition is to specify the value of ϕ on the boundary of the outside region, ie

$$\phi(R) = \phi_0$$

$$\Rightarrow \frac{C_0}{R} = \phi_0 , \quad C_0 = \phi_0 R$$

$$\phi(r) = \phi_0 \frac{R}{r}$$

(from preceeding solution, we know that charging the sphere to voltage ϕ_0 (statvolts) induces a net charge $q = \phi_0 R$ on it)

These two versions of the conducting sphere problem are examples of a more general <u>boundary value problem</u>

Solve $\nabla^2\phi = 0$ in a given region of space subject to one of the following two types of boundary conditions on the bounding surfaces of the region

i) <u>Neumann</u> boundary condition

$\frac{\partial\phi}{\partial n}$ - normal derivative of ϕ is specified on the bounding surface

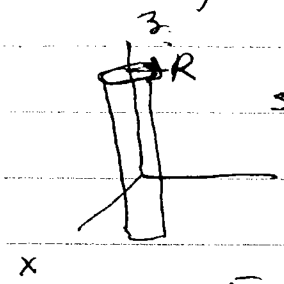
ii) <u>Dirichlet</u> boundary condition

ϕ - value of ϕ is specified on the bounding surfaces

If the bounding surfaces consist of <u>disjoint</u> pieces, it is possible to specify either (i) or (ii) on each piece separately to get a mixed boundary value problem.

Some more problems

infinite conducting wire of radius R with line charge density λ = charge per unit length



surface charge $\sigma = \frac{\lambda}{2\pi R}$

Expect cylindrical symmetry $\Rightarrow \phi$ depends only on cylindrical coord r.

$\nabla^2\phi = 0$ for $r > R$, $r < R$

use ∇^2 in cylindrical coords – only radial term non vanishing

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

$$r\frac{d\phi}{dr} = C_0 \quad \text{constant}$$

$$\frac{d\phi}{dr} = \frac{C_0}{r}$$

$$\phi(r) = C_0 \ln r + C_1 \quad \text{const}$$

note: one cannot now choose $\phi \to 0$ as $r \to \infty$!

one needs to fix zero of ϕ at some other radius. a convenient choice is $r = R$, but any other choice could also be made.

$$\phi^{out} = C_0^{out} \ln r + C_1^{out}$$
$$\phi^{in} = C_0^{in} \ln r + C_1^{in}$$

ϕ^{in} = const in conductor $\rightarrow C_0^{in} = 0$
or ϕ^{in} should not diverge as $r \to 0 \Rightarrow C_0^{in} = 0$

So $\phi^{in} = C_1^{in}$ constant

boundary condition at $r = R$

$$\left[-\frac{d\phi^{out}}{dr} + \frac{d\phi^{in}}{dr} \right]_{r=R} = 4\pi\sigma$$

$$\Rightarrow \quad -\frac{C_0^{out}}{R} = 4\pi\sigma = 4\pi\left(\frac{\lambda}{2\pi R}\right) = \frac{2\lambda}{R}$$
$$C_0^{out} = -2\lambda$$

$$\phi^{out}(r) = -2\lambda \ln r + C_1^{out}$$

continuity of ϕ

$$\phi^{in}(R) = \phi^{out}(R) \quad \Rightarrow \quad C_1^{in} = -2\lambda \ln R + C_1^{out}$$

Remaining const C_1^{out} is not too important as it is just a common additive constant to both ϕ^{in} and ϕ^{out} $\rightarrow$ does not change $\vec{E} = -\vec{\nabla}\phi$.

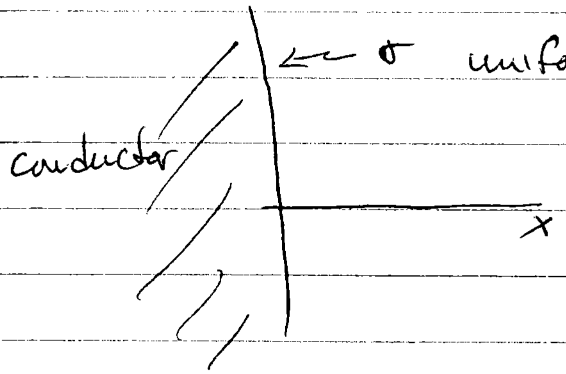
If use the condition $\phi(R) = 0$ then we can solve for C_1^{out}.

$$0 = -2\lambda \ln R + C_1^{out} \quad \Rightarrow \quad C_1^{out} = 2\lambda \ln R$$

$$\Rightarrow \quad \phi(r) = \begin{cases} -2\lambda \ln(r/R) & r > R \\ 0 & r < R \end{cases}$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \begin{cases} \frac{2\lambda}{r}\,\hat{r} & r > R \\ 0 & r < R \end{cases}$$

infinite conducting half space



$\leftarrow \sigma$ uniform surface charge density

expect ϕ depends only on x

$$\nabla^2 \phi = \frac{d^2\phi}{dx^2} = 0$$

$$\rightarrow \begin{cases} \phi^>(x) = c_0^> x + c_1^> & x > 0 \\ \phi^<(x) = c_0^< x + c_1^< & x < 0 \end{cases}$$

for $x < 0$, $\phi = \text{const}$ in conductor $\Rightarrow c_0^< = 0$

at $x = 0$, ϕ continuous $\Rightarrow \phi^<(0) = \phi^>(0)$

$$c_1^< = c_1^>$$

$\frac{d\phi}{dx}$ discontinuous $\Rightarrow$

$$-\left.\frac{d\phi^>}{dx}\right|_{x=0} = 4\pi\sigma$$

$$c_0^> = -4\pi\sigma$$

$$\Rightarrow \quad \phi(x) = \begin{cases} -4\pi\sigma x + c_1^> & x > 0 \\ c_1^> & x < 0 \end{cases}$$

const $c_1^>$ does not change value of $\vec{E}$

as for the wire, we cannot choose $\phi \to 0$ as $x \to \infty$.
~~we can set $\phi = 0$ at~~

$$-\vec{\nabla}\phi = \vec{E} = \begin{cases} 4\pi\sigma\,\hat{x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

<u>infinite charged plane</u>

similar to previous problem, but now no conductor at $x < 0$, just free space on both sides of the charged plane at $x = 0$.

~~expect ϕ depends only on $|x|$ by symmetry~~

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = 0 \quad \Rightarrow \quad \begin{aligned} \phi^> &= c_0^> x + c_1^> & x > 0 \\ \phi^< &= c_0^< x + c_1^< & x < 0 \end{aligned}$$

continuity of ϕ at $x = 0$

$$\rightarrow \quad \phi^>(0) = \phi^<(0) \quad \Rightarrow \quad c_1^> = c_1^<$$

discontinuity of $d\phi/dx$ at $x = 0$

$$-\frac{d\phi^>}{dx} + \frac{d\phi^<}{dx} = 4\pi\sigma$$

$$-c_0^> + c_0^< = 4\pi\sigma$$

Define $\bar{c}_0 = \dfrac{c_0^> + c_0^<}{2}$

Then we can write

$$C_0^< = \bar{C}_0 + 2\pi\sigma$$
$$C_0^> = \bar{C}_0 - 2\pi\sigma$$

$$\phi = \begin{cases} -2\pi\sigma x + \bar{C}_0 x + C_1^> & x > 0 \\ 2\pi\sigma x + \bar{C}_0 x + C_1^> & x < 0 \end{cases}$$

$$\Rightarrow -\frac{d\phi}{dx} = \vec{E} = \begin{cases} (2\pi\sigma - \bar{C}_0)\,\hat{x} & x > 0 \\ (-2\pi\sigma - \bar{C}_0)\,\hat{x} & x < 0 \end{cases}$$

Const $C_1^>$ does not effect $\vec{E}$ – additive const to ϕ

$\bar{C}_0$ represents const uniform electric field $-\bar{C}_0\hat{x}$, that exists independently of the charged surface

If we assumed that all $\vec{E}$ fields are just those arising from the plane, then we can set $\bar{C}_0 = 0$. Equivalently, if the plane is the only source of $\vec{E}$, then we expect ϕ depends only on $|x|$ by symmetry. $\Rightarrow$ $C_0^< = -C_0^>$ and again $\bar{C}_0 = 0$. In this case

$$\phi(x) = \begin{cases} -2\pi\sigma x & x > 0 \\ 2\pi\sigma x & x < 0 \end{cases}$$

(we also set $C_1^> = 0$ here corresponding to $\phi(0) = 0$)

$$\vec{E}(x) = \begin{cases} 2\pi\sigma\,\hat{x} & x > 0 \\ -2\pi\sigma\,\hat{x} & x < 0 \end{cases}$$

$\vec{E}$ is constant by oppositely directed on either side of the charged plane