

# Green's Theorem, Uniqueness, Green function - part II

We want to show that the boundary value problem we described is well posed - i.e. there is a unique solution. We start by deriving Green's Theorems

Consider  $\int_V d^3r \nabla \cdot \vec{A} = \oint_S da \hat{m} \cdot \vec{A}$  Gauss theorem

let  $\vec{A} = \phi \vec{\nabla} \psi$   $\phi, \psi$  any two scalar functions

$$\Rightarrow \nabla \cdot \vec{A} = \phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi$$

$$\phi \vec{\nabla} \psi \cdot \hat{m} = \phi \frac{\partial \psi}{\partial m}$$

$$\Rightarrow \int_V d^3r (\phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi) = \oint_S da \phi \frac{\partial \psi}{\partial m} \quad \left. \vphantom{\int_V} \right\} \text{Green's 1st identity}$$

let  $\phi \leftrightarrow \psi$

$$\int_V d^3r (\psi \nabla^2 \phi + \vec{\nabla} \psi \cdot \vec{\nabla} \phi) = \oint_S da \psi \frac{\partial \phi}{\partial m}$$

subtract

$$\int_V d^3r (\phi \nabla^2 \psi - \psi \nabla^2 \phi) = \oint_S da \left( \phi \frac{\partial \psi}{\partial m} - \psi \frac{\partial \phi}{\partial m} \right) \quad \left. \vphantom{\int_V} \right\} \text{Green's 2nd identity}$$

Apply Green's 2nd identity with  $\psi = \frac{1}{|\vec{r} - \vec{r}'|}$   
 $\vec{r}'$  is integration variable,  $\phi$  is the scalar potential  
 with  $\nabla^2 \phi = -4\pi\rho$ . Use  $\nabla^2 \psi = \nabla'^2 \psi = -4\pi \delta(\vec{r} - \vec{r}')$

$$\int_V d^3r' \left[ \phi(\vec{r}') [-4\pi \delta(\vec{r} - \vec{r}')] - \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) (-4\pi \rho(\vec{r}')) \right]$$

$$= \oint_S da' \left[ \phi \frac{\partial}{\partial m'} \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) - \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \phi}{\partial m'} \right]$$

If  $\vec{r}$  lies within the volume  $V$ , then

$$(*) \quad \phi(\vec{r}) = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S da' \left[ \frac{1}{4\pi} \frac{\partial \phi}{\partial m'} - \phi \frac{\partial}{\partial m'} \left( \frac{1}{|\vec{r}-\vec{r}'|} \right) \right]$$

Note: if  $\vec{r}$  lies outside the volume  $V$ , then

$$0 = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S da' \left[ \frac{1}{4\pi} \frac{\partial \phi}{\partial m'} - \phi \frac{\partial}{\partial m'} \left( \frac{1}{|\vec{r}-\vec{r}'|} \right) \right]$$

↑  
potential from a  
surface charge density

$$\sigma = \frac{1}{4\pi} \frac{\partial \phi}{\partial m'}$$

↑  
potential from a  
surface dipole layer of  
dipole strength density

$$\frac{\phi}{4\pi}$$

From (\*), if  $S \rightarrow \infty$  and  $E \sim \frac{\partial \phi}{\partial m} \rightarrow 0$  faster than  $\frac{1}{r}$ , then the surface integral vanishes and we recover Coulomb's law  $\phi(\vec{r}) = \int_V d^3r' \rho(\vec{r}')/|\vec{r}-\vec{r}'|$

(\*) gives the generalization of Coulomb's law to a system with a finite boundary

For a charge free volume  $V$ , i.e.  $\rho(r) = 0$  in  $V$ , the potential everywhere is determined by the potential and its normal derivative on the surface.

But one cannot in general freely specify both  $\phi$  and  $\frac{\partial \phi}{\partial m}$  on the boundary surface since the resulting  $\phi$  from (\*) would not in general obey Laplace's equation  $\nabla^2 \phi = 0$ .

Specifying both  $\phi$  and  $\frac{\partial\phi}{\partial n}$  on surface is known as "Cauchy" boundary conditions — for Laplace's eqn, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

## Uniqueness

If we have a system of charges in vol  $V$ , and either the potential  $\phi$ , or its normal derivative  $\frac{\partial\phi}{\partial n}$ , is specified on the surfaces of  $V$ , then there is a unique solution to Poisson's equation inside  $V$ . Specifying  $\phi$  is known as Dirichlet boundary conditions. Specifying  $\frac{\partial\phi}{\partial n}$  is known as Neumann boundary conditions.

proof: Suppose we had two solutions  $\phi_1$  and  $\phi_2$ , both with  $-\nabla^2\phi = 4\pi\rho$  inside  $V$ , and obeying specified b.c. on surface of  $V$ .

$$\text{Define } U = \phi_2 - \phi_1 \rightarrow \nabla^2 U = 0 \text{ inside } V$$

and  $U = 0$  on surface  $S$  — for Dirichlet b.c.

or  $\frac{\partial U}{\partial n} = 0$  on surface  $S$  — for Neumann b.c.

Use Green's 1<sup>st</sup> identity with  $\phi = \psi = U$

$$\int_V d^3r (U \nabla^2 U + \vec{\nabla} U \cdot \vec{\nabla} U) = \oint_S da U \frac{\partial U}{\partial n}$$

$\underset{\text{as } \nabla^2 U = 0}{\overset{0}{U \nabla^2 U}}$ 
 $\quad \quad \quad \underset{\text{as } U \text{ or } \frac{\partial U}{\partial n} = 0}{\overset{0}{\oint_S da U \frac{\partial U}{\partial n}}}$

$$\Rightarrow \int_V d^3r |\vec{\nabla} u|^2 = 0 \quad \Rightarrow \vec{\nabla} u = 0$$

$$\Rightarrow u = \text{const}$$

For Dirichlet b.c.,  $u = 0$  on surface  $S$ , so  $\text{const} = 0$  and  $\phi_1 = \phi_2$ . Solution is unique.

For Neumann b.c.,  $\phi_1$  and  $\phi_2$  differ only by an arbitrary constant. Since  $\vec{E} = -\vec{\nabla}\phi$ , the electric fields  $\vec{E}_1 = -\vec{\nabla}\phi_1$  and  $\vec{E}_2 = -\vec{\nabla}\phi_2$  are the same.

~~Problem~~ If boundary ~~surface~~ surface  $S$  consists of several disjoint pieces, then solution is unique if specify  $\phi$  on some pieces and  $\frac{\partial\phi}{\partial n}$  on other pieces.

Solution of Poisson's equation with both  $\phi$  and  $\frac{\partial\phi}{\partial n}$  specified on the same surface  $S$  (Cauchy b.c.) does not in general exist, since specifying either  $\phi$  or  $\frac{\partial\phi}{\partial n}$  alone is enough to give a unique solution.

## Green's function - part II

Green's 2<sup>nd</sup> identity

$$\int_V d^3r' (\phi \nabla'^2 \psi - \psi \nabla'^2 \phi) = \oint_S da' \left( \phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right)$$

Apply above with  $\begin{cases} \phi(\vec{r}') \text{ electrostatic potential with } \nabla'^2 \phi = -4\pi\rho(\vec{r}') \\ \psi(\vec{r}') = G(\vec{r}, \vec{r}') \text{ the Green function satisfying} \end{cases}$

$$\nabla'^2 G(\vec{r}, \vec{r}') = -4\pi\delta(\vec{r} - \vec{r}')$$

We saw one solution of above is  $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$   
but a more general solution is

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}')$$

where  $\nabla'^2 F(\vec{r}, \vec{r}') = 0$ , for  $\vec{r}'$  in volume  $V$

we will choose  $F(\vec{r}, \vec{r}')$  to simplify solution of  $\phi$

$$\Rightarrow \int_V d^3r' (\phi(\vec{r}') \nabla'^2 G(\vec{r}, \vec{r}') - G(\vec{r}, \vec{r}') \nabla'^2 \phi(\vec{r}'))$$

$$= \int_V d^3r' (\phi(\vec{r}') [-4\pi\delta(\vec{r} - \vec{r}')] - G(\vec{r}, \vec{r}') [-4\pi\rho(\vec{r}')] )$$

$$= -4\pi\phi(\vec{r}) + 4\pi \int_V d^3r' G(\vec{r}, \vec{r}') \rho(\vec{r}')$$

$$= \oint_S da' \left( \phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

$$\phi(\vec{r}) = \int_V d^3r' G(\vec{r}, \vec{r}') \rho(\vec{r}') + \oint_S \frac{da'}{4\pi} \left( G(\vec{r}, \vec{r}') \frac{\partial \phi(\vec{r}')}{\partial n'} - \phi(\vec{r}') \frac{\partial G(\vec{r}, \vec{r}')}{\partial n'} \right)$$

Consider Dirichlet boundary problem. If we can choose  $F(\vec{r}, \vec{r}')$  such that  $G_D(\vec{r}, \vec{r}') = 0$  for  $\vec{r}'$  on the boundary surface  $S$ , then above simplifies to

$$\left[ \phi(\vec{r}) = \int_V d^3r' G_D(\vec{r}, \vec{r}') \rho(\vec{r}') - \oint_S \frac{da'}{4\pi} \phi(\vec{r}') \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'} \right]$$

Since  $\rho(\vec{r})$  is specified in  $V$ , and  $\phi(\vec{r})$  is specified on  $S$ , above then gives desired solution for  $\phi(\vec{r})$  inside volume  $V$ .

Finding  $G_D$  is therefore equivalent to finding an  $F(\vec{r}, \vec{r}')$  such that ~~the~~  $\nabla'^2 F(\vec{r}, \vec{r}') = 0$  for  $\vec{r}'$  in  $V$  (solves Laplace eqn) and

$$F(\vec{r}, \vec{r}') = \frac{-1}{|\vec{r} - \vec{r}'|} \quad \text{for } \vec{r}' \text{ on boundary surface } S'$$

Always exists unique solution for  $F$

Next consider Neumann boundary problem.

One might think to find  $F(\vec{r}, \vec{r}')$  such that  $\frac{\partial G(\vec{r}, \vec{r}')}{\partial m'} = 0$  on boundary surface. But this is not possible.

$$\begin{aligned} \text{Consider } \int_V \nabla'^2 G(\vec{r}, \vec{r}') d^3 r' &= \int_V \vec{\nabla}' \cdot \vec{\nabla}' G(\vec{r}, \vec{r}') d^3 r' \\ &= \oint_S \vec{\nabla}' G(\vec{r}, \vec{r}') \cdot \hat{m} da' \\ &= \oint_S \frac{\partial G(\vec{r}, \vec{r}')}{\partial m'} da' = -4\pi \quad \text{since} \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \nabla'^2 G = -4\pi \delta(\vec{r} - \vec{r}') \end{aligned}$$

So we can't have  $\frac{\partial G}{\partial m'} = 0$  for  $\vec{r}'$  on  $S$

Simplest choice is then  $\frac{\partial G_N(\vec{r}, \vec{r}')}{\partial m'} = \frac{-4\pi}{S}$  for  $\vec{r}'$  on  $S$   
 $S$  ← area of surface

Then

$$\begin{aligned} \phi(\vec{r}) &= \int_V d^3 r' G_N(\vec{r}, \vec{r}') \rho(\vec{r}') + \oint \frac{da'}{4\pi} G(\vec{r}, \vec{r}') \frac{\partial \phi(\vec{r}')}{\partial m'} \\ & \qquad \qquad \qquad + \oint \frac{da'}{4\pi} \phi(\vec{r}') \left( \frac{-4\pi}{S} \right) \end{aligned}$$

$$\left[ \phi(\vec{r}') = \int_V d^3 r' G_N(\vec{r}, \vec{r}') \rho(\vec{r}') + \oint \frac{da'}{4\pi} G(\vec{r}, \vec{r}') \frac{\partial \phi(\vec{r}')}{\partial m'} \right]$$

+  $\langle \phi \rangle_S$

Since  $\rho(r)$  is specified in  $V$   
 and  $\frac{\partial \phi}{\partial m}$  is specified on  $S'$

↑ constant = average value of  $\phi$  on surface  $S'$ .

Above gives solution  $\phi(r)$  in  $V$  within additive constant  $\langle \phi \rangle_S$   
 Since  $\vec{E} = -\vec{\nabla} \phi$ , the const  $\langle \phi \rangle_S$  is of no consequence.

Finding  $G_N(\vec{r}, \vec{r}')$  is therefore equivalent to finding an  $F(\vec{r}, \vec{r}')$  such that

$$\nabla'^2 F(\vec{r}, \vec{r}') = 0 \text{ for } \vec{r}' \text{ in } V$$

$$\text{and } \frac{\partial F(\vec{r}, \vec{r}')}{\partial n'} = \frac{-4\pi}{S'} \text{ for } \vec{r}' \text{ on surface } S'$$

always exists a unique solution (within additive constant)

While  $G_D$  and  $G_N$  always exist in principle, they depend in detail on the shape of the surface  $S'$  and are difficult to find except for simple geometries

In preceding we defined  $G$  by  $\nabla'^2 G(\vec{r}, \vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$

But our earlier interpretation of  $G(\vec{r}, \vec{r}')$  was that it was potential at  $\vec{r}$  due to point source at  $\vec{r}'$ , i.e.  $\nabla^2 G(\vec{r}, \vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$ . Note, for general surface  $S'$ ,  $G(\vec{r}, \vec{r}')$  is not in general a function of  $|\vec{r}-\vec{r}'|$  but depends on  $\vec{r}$  and  $\vec{r}'$  separately. But the equivalence of the two definitions of  $G$  above is obtained by noting that one can prove the symmetry property  $G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c.

(see Jackson, end section 1.10)