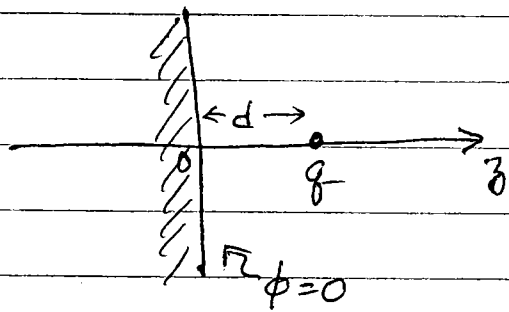


Image Charge method

For single geometries, can try to obtain G_D or G_N by placing a set of "image charges" outside the volume of interest V , i.e. on the "other side" of the system boundary surface S . Because these image charges are outside V , their contrib to the potential inside V obeys $\nabla^2 \phi^{\text{image}} = 0$, as necessary. Choose location of image charges so that total ϕ has desired boundary condition.

1) Charge in front of infinite grounded plane



$$\text{want } \nabla^2 \phi = -4\pi q \delta(x) \delta(y) \delta(z-d)$$
$$\phi = 0 \text{ for } z=0$$

If we find a solution to above it is the unique solution

Solution - put fictitious image charge $-q$ at $z = -d$
 ϕ is Coulomb potential from the real charge + the image

$$\phi(\vec{r}) = \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}}$$

real charge image charge

above satisfies $\phi(x, y, 0) = 0$ as required

$$\text{Also, } \nabla^2 \phi = -4\pi q \delta(\vec{r} - d\hat{z}) + 4\pi q \delta(\vec{r} + d\hat{z})$$

$$= -4\pi q \delta(\vec{r} - d\hat{z}) \text{ for region } z > 0$$

Can now find \vec{E} for $z > 0$

$$\vec{E} = -\vec{\nabla}\phi$$

$$\text{In particular } E_z = -\frac{\partial\phi}{\partial z} = +q \left[\left(\frac{1}{2}\right) \frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \left(\frac{1}{2}\right) \frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right]$$

$$E_z = q \left[\frac{(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right]$$

We can use above to compute the surface charge density $\sigma(x,y)$ induced on the surface of the conducting plane. At conductor surface

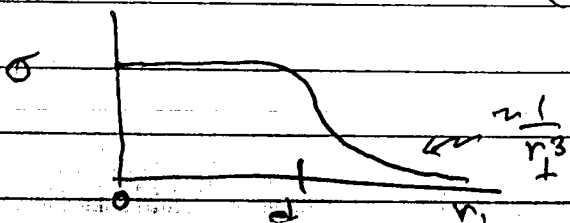
$$-\frac{\partial\phi}{\partial n} = 4\pi\sigma$$

$$\Rightarrow \sigma = -\frac{1}{4\pi} \frac{\partial\phi}{\partial z} = \frac{1}{4\pi} E_z(x,y,z=0)$$

$$\sigma(x,y) = \frac{q}{4\pi} \left[\frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}} \right]$$

$$= -\frac{q}{2\pi} \frac{d}{(x^2+y^2+d^2)^{3/2}} = -\frac{qd}{2\pi (r_{\perp}^2+d^2)^{3/2}}$$

$$r_{\perp} = \sqrt{x^2+y^2}$$



Total induced charge is

$$q^{\text{induced}} = \int_{-\infty}^{\infty} dx dy \sigma(x, y)$$

$$= 2\pi \int_0^{\infty} dr_{\perp} r_{\perp} \sigma(r_{\perp})$$

$$= 2\pi \int_0^{\infty} dr_{\perp} \frac{r_{\perp} (-qd)}{2\pi (r_{\perp}^2 + d^2)^{3/2}}$$

$$= -qd \left[\frac{-1}{(r_{\perp}^2 + d^2)^{1/2}} \right]_0^{\infty}$$

$$= -qd \left[0 - \frac{-1}{d} \right]$$

$$q^{\text{induced}} = -q \quad \text{induced charge} = \text{image charge}$$

Force on charge q in front of conducting plane is due to the induced σ . The E field of this σ is, for $z > 0$, the same as the E field of the image charge.

$$\Rightarrow \vec{F} = \frac{-q^2}{(2d)^2} \hat{z} = \frac{-q^2}{4d^2} \hat{z} \quad \text{attractive}$$

Work done to move q into position from infinity is

$$W = - \int_{\infty}^d d\vec{l} \cdot \vec{F} = - \int_{\infty}^d dz F_z$$

we must oppose electrostatic force \vec{F}

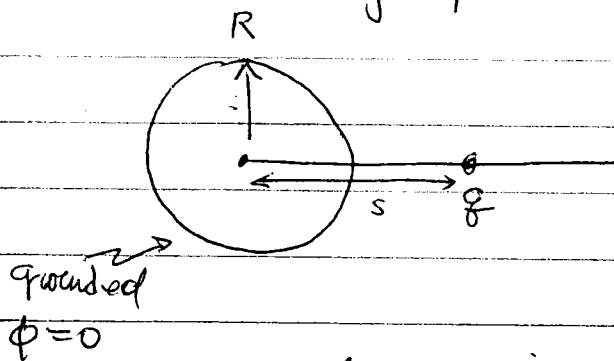
$$W = \int_d^{\infty} dz \left(\frac{-q^2}{4z^2} \right) = \frac{-q^2}{4d}$$

$W < 0 \Rightarrow$ energy released

Note: W above is not the electrostatic energy that would be present if the image charge were real i.e. it is not $\phi^{\text{image}}(\vec{r} = d\hat{z}) = \frac{-q^2}{2d}$

One way to see why is to note that as q is moved quasistatically in towards the conducting plane, the image charge also must be moving to stay equidistant on the opposite side,

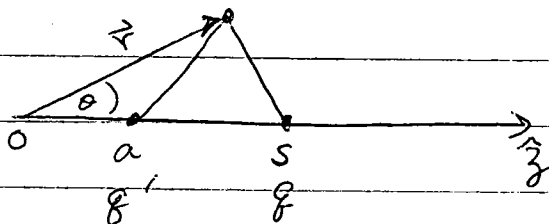
2) point charge in front of a grounded ($\phi=0$) conducting sphere.



charge q placed a distance s from center of grounded conducting sphere of radius R

place image charge q' inside sphere so that the combined ϕ from q and q' vanishes on surface of sphere.

By symmetry, q' should lie on the same radial line as q does. call the distance of q' from the origin " a "



potential at position \vec{r} is

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - s\hat{z}|} + \frac{q'}{|\vec{r} - a\hat{z}|}$$

$$= \frac{q}{(r^2 + s^2 - 2sr\cos\theta)^{1/2}} + \frac{q'}{(r^2 + a^2 - 2ra\cos\theta)^{1/2}}$$

Can we choose q' and a so that $\phi(r, \theta) = 0$ for all θ ?

$$\phi(R, \theta) = \frac{q}{(R^2 + s^2 - 2sR \cos \theta)^{1/2}} + \frac{q'}{(R^2 + a^2 - 2aR \cos \theta)^{1/2}}$$

make denominators look alike

$$R^2 + a^2 - 2aR \cos \theta = \frac{a}{s} \left(\frac{s}{a} R^2 + sa - 2sR \cos \theta \right)$$

if choose $sa = R^2$, i.e. $a = R^2/s$, then $\frac{sR^2}{a} = s^2$
and then the denominator of the 2nd term is

$$\left[\frac{R^2}{s^2} (s^2 + R^2 - 2sR \cos \theta) \right]^{1/2} = \frac{R}{s} [s^2 + R^2 - 2sR \cos \theta]^{1/2}$$

$$\Rightarrow \phi(R, \theta) = \frac{q}{(R^2 + s^2 - 2sR \cos \theta)^{1/2}} + \frac{q'(s/R)}{(R^2 + s^2 - 2sR \cos \theta)^{1/2}}$$

So choose $q'(s/R) = -q \Rightarrow q' = -qR/s$
to get $\phi(R, \theta) = 0$

Solution is

$$\begin{aligned} \phi(r, \theta) &= \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{qR/s}{\left(r^2 + \frac{R^4}{s^2} - 2r \frac{R^2}{s} \cos \theta \right)^{1/2}} \\ &= \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q}{\left(\frac{s^2 r^2}{R^2} + R^2 - 2rs \cos \theta \right)^{1/2}} \end{aligned}$$

Can get induced surface charge on sphere by

$$4\pi \sigma = \vec{E} \cdot \hat{r} = - \left. \frac{\partial \phi}{\partial r} \right|_{r=R} \quad \text{see Jackson Eq (2.5) for result}$$

$$\sigma(\theta) = -\frac{q}{4\pi R S} \frac{1 - (R/S)^2}{(1 + (R/S)^2 - 2(R/S)\cos\theta)^{3/2}}$$

$\sigma(\theta)$ is greatest at $\theta=0$, as one should expect

Can integrate $\sigma(\theta)$ to get total induced charge. One finds

$$2\pi \int_0^\pi d\theta \sin\theta R^2 \sigma(\theta) = q' = -qR/S$$

In general, total induced charge = sum of all image charges

Force of attraction of charge to sphere

Force on q is due to electric field from induced charge σ which is the same as the electric field from the image charge q' .

$$\vec{F} = \frac{q q' \hat{z}}{(s-a)^2} = \frac{-q^2 (R/S) \hat{z}}{(s - R^2/S)^2} = \frac{-q^2 R S \hat{z}}{(s^2 - R^2)^2}$$

Close to the surface of the sphere, $s \approx R$, so write $s = R + d$ where $d \ll R$. Then

$$\vec{F} = \frac{-q^2 R S}{(s-R)^2 (s+R)^2} = \frac{-q^2 R (R+d)}{d^2 (2R+d)^2} \approx \frac{-q^2}{4d^2}$$

get same result as for infinite flat grounded plane. When q is so close to surface that $d \ll R$, the charge does not "see" the curvature of the surface.

far from the surface, $s \gg R$

$$\vec{F} = \frac{q q' \hat{z}}{(s-a)^2} = \frac{-q^2 R s}{(s^2 - R^2)^2} \hat{z} \approx \frac{-q^2 R}{s^3} \hat{z}$$

$F \sim \frac{1}{s^3}$ very different from flat plane
also different from point charge

Note: In preceding two problems, what we found was a ϕ such that $\nabla^2 \left(\frac{\phi}{q} \right) = -4\pi \delta(\vec{r} - \vec{r}_0)$, for a charge at \vec{r}_0 , and $\phi = 0$ on the boundary. Such a ϕ is nothing more than G_0 the corresponding Green function for Dirichlet boundary conditions.

Suppose now that instead of a grounded sphere we have a sphere with fixed net charge Q .

We want to add new image charge to represent this case.

If we put $q' = -qR/s$ at $a = R/s$ as before, the boundary condition of $\phi = \text{const}$ on surface $r = R$ is met, but the net charge on the sphere is q' (the induced charge) not the desired Q . We therefore need to add new image charge(s) of total charge $Q - q'$ (so total image charge is Q) in such a way that we keep ϕ constant on the surface of the sphere. The way to do this is to put $Q - q'$ at the origin!

Solution is

$$\phi(r, \theta) = \frac{Q + qR/s}{r} + \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q}{\left(\frac{s^2 r^2 + R^2 - 2rs \cos \theta}{R^2}\right)^{1/2}}$$

The force on the charge q is due to the \vec{E} field of the images

$$\vec{F} = F \hat{z} = \frac{q(Q + qR/s)}{s^2} \hat{z} + \frac{q q'}{(s-a)^2} \hat{z}$$

$$F = \frac{qQ}{s^2} + \frac{q^2 R/s}{s^2} - \frac{q^2 R/s}{(s - R^2/s)^2}$$

$$= \frac{qQ}{s^2} + q^2 R \left[\frac{1}{s^3} - \frac{1}{s^3 (1 - R^2/s^2)^2} \right]$$

$$= \frac{qQ}{s^2} + \frac{q^2 R}{s^3} \left[1 - \frac{1}{(1 - R^2/s^2)^2} \right]$$

$$F = \frac{qQ}{s^2} - \frac{q^2 R^3}{s} \frac{2 - R^2/s^2}{(s^2 - R^2)^2}$$

For large $s \gg R$ far from surface

$$F \approx \frac{qQ}{s^2} - \frac{2q^2 R^3}{s^5}$$

leading term is just
Coulomb force between q
and Q at origin

for $Q > 0$, F is always repulsive for large enough s .

For $s = R + d$, $d \ll R$ close to surface

$$F = \frac{qQ}{(R+d)^2} - \frac{q^2 R^3}{(R+d)} \frac{2 - \frac{R^2}{(R+d)^2}}{(R^2 + d^2 + 2Rd - R^2)^2}$$

$$\approx \frac{qQ}{R^2} - \frac{q^2 R^3}{R} \frac{(2-1)}{4R^2 d^2}$$

$$F \approx \frac{qQ}{R^2} - \frac{q^2}{4d^2} \approx -\frac{q^2}{4d^2} \quad \text{for } d \text{ small enough}$$

F is always attractive for small enough d , and is equal to the force in front of a grounded plane, no matter what is the value of Q ! This is because the image charge q' lies so much closer to q than does the $Q-q'$ at the origin, that it dominates the force.

The cross over from attractive to repulsive occurs at a distance s that depends on Q . This distance is given by

$$\frac{Q}{q} = \frac{R^3 s (2 - R^2/s^2)}{(s^2 - R^2)^2} = \left(\frac{R^3}{s}\right) \frac{2 - (R/s)^2}{[1 - (R/s)^2]^2}$$

let $x = R/s \in (0, 1)$

$$\frac{Q}{q} = x^3 \frac{(2-x^2)}{(1-x^2)^2}$$

gives 5th order polynomial in x
no analytic solution
can solve graphically

For $\frac{Q}{g} = 1$, cross over is at $\frac{R}{S} = 0.62$

$$S = 1.6R$$

$\frac{Q}{g} = 0.1$ cross over is at $\frac{R}{S} \approx 0.36$

$$S = 2.8R$$