

# Spherical Coordinates

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} =$$

$$\phi(r, \theta, \varphi) = R(r) \Theta(\theta) \Phi(\varphi)$$

$$r^2 \nabla^2 \phi = \Theta \Phi \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R \Phi}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{R \Theta}{\sin^2 \theta} \frac{d^2 \Phi}{d\varphi^2} =$$

$$\frac{r^2 \sin^2 \theta}{\Phi} \nabla^2 \phi = \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = C$$

depends only on  $r$  and  $\theta$

= -const

depends only on  $\varphi$

= const

take  $\frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = -m^2$

$$\Rightarrow \boxed{\Phi = e^{\pm i m \varphi}}$$

$m$  integer for  $2\pi$  periodicity in  $\varphi$

$$\Rightarrow \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = m^2$$

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0$$

depends only on  $r$

= const

depends only on  $\theta$

= -const

call the const  $l(l+1)$

For R

$$\frac{1}{r} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - l(l+1) = 0$$

Solutions are of the form  $R(r) = a_l r^l + b_l r^{-(l+1)}$   
substitute in to verify

$$\begin{aligned} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) &= \frac{d}{dr} \left( r^2 (la_l r^{l-1} - (l+1)b_l r^{-l-2}) \right) \\ &= \frac{d}{dr} \left( la_l r^{l+1} - (l+1)b_l r^{-l} \right) \\ &= l(l+1)a_l r^l + l(l+1)b_l r^{-(l+1)} = l(l+1)R \end{aligned}$$

For  $\Theta$  :

$$\frac{1}{\sin^2 \theta} \frac{d}{d\theta} \left( \sin^2 \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = -l(l+1)$$

let  $x = \cos \theta$

$$dx = -\sin \theta d\theta$$

$$d\theta = \frac{-dx}{\sin \theta}$$

$$0 \leq \theta \leq \pi$$

above becomes

solutions for  $-1 \leq x \leq 1$   
correspond to  $l \geq 0$  integers

$$\frac{d}{dx} \left[ (1-x^2) \frac{d\Theta}{dx} \right] + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] \Theta = 0$$

Called generalized Legendre Equation - solutions are called the associated Legendre functions.

ordinary Legendre polynomials are solutions  
for  $m=0$

For the special case  $m=0$ , i.e. the solution has azimuthal symmetry and  $\Phi$  does not depend on the angle  $\varphi$  (i.e. rotational symmetry about  $\hat{z}$  axis),

We want the solutions to

$$\frac{d}{dx} \left[ (1-x^2) \frac{d\Phi}{dx} \right] + \ell(\ell+1) \Phi = 0$$

The solutions are known as the Legendre polynomials,  $P_\ell(x)$ .

They are given, for  $\ell$  integer, by

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left( \frac{d}{dx} \right)^\ell (x^2-1)^\ell \quad \text{Rodriguez's formula}$$

The lowest  $\ell$  polynomials are

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

In general,  $P_\ell(x)$  is a polynomial of order  $\ell$  with only even powers if  $\ell$  is even, and only odd powers if  $\ell$  is odd.  $\Rightarrow P_\ell(x) \begin{cases} \text{even in } x & \text{for } \ell \text{ even} \\ \text{odd in } x & \text{for } \ell \text{ odd} \end{cases}$

$P_\ell(x)$  is normalized so that  $P_\ell(1) = 1$

Note: Legendre polynomials are only for integer  $l \geq 0$ .

What about solutions for non integer  $l$ ?

The  $P_l(x)$  give one solution for each integer  $l$ .

But  $P_l(x)$  are defined by a 2<sup>nd</sup> order differential equation - shouldn't there be a 2<sup>nd</sup> independent solution for each  $l$ ?

It turns out that these "2<sup>nd</sup>" solutions, as well as solutions for non integer  $l$ , all blow up at either  $x = -1$  or  $x = 1$ , i.e. at  $\theta = 0$  or  $\theta = \pi$ .

They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval  $-1 \leq x \leq 1$ .

$$\int_{-1}^1 dx P_l(x) P_m(x) = \int_0^\pi d\theta \sin\theta P_l(\cos\theta) P_m(\cos\theta) = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases}$$

$\Rightarrow$  we can expand any function  $f(\theta)$ ,  $0 \leq \theta \leq \pi$ , as a linear combination of the  $P_l(\cos\theta)$ .

This is the reason they are useful for solving problems of Laplace's eqn with spherical boundary surfaces.

For  $m \neq 0$ , the solutions to (see Jackson 3.5)

$$\frac{d}{dx} \left[ (1-x^2) \frac{d\Theta}{dx} \right] + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] \Theta = 0$$

are the associated Legendre functions  $P_l^m(x)$ .

For  $P_l^m(x)$  to be finite in interval  $-1 \leq x \leq 1$

one again finds that  $l$  must be integer  $l \geq 0$ , and integer  $m$  must satisfy  $|m| \leq l$ , i.e.  $m = -l, -(l-1), \dots, 0, \dots, (l-1), l$

For each  $l$  and  $m$  there is only one such non divergent solution.

It is typical to combine the solutions  $P_l^m(\cos\theta)$  to the  $\theta$ -part of the equation with the  $\Phi_m(\varphi) = e^{im\varphi}$  solutions to the  $\varphi$ -part of the equation to define the spherical harmonics

$$Y_{lm}(\theta, \varphi) \equiv \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$$

The  $Y_{lm}$  are orthogonal

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \delta_{ll'} \delta_{mm'}$$

and are a complete set of basis functions for expanding any function  $f(\theta, \varphi)$  defined on the surface of a sphere.

## Examples with azimuthal symmetry $m=0$

General solution to  $\nabla^2\phi=0$  can be written in form

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta)$$

determine the  $A_l$  and  $B_l$  from the boundary conditions of the particular problem.

① Suppose one is given  $\phi(R,\theta) = \phi_0(\theta)$  on surface of sphere of radius  $R$ .

To find solution of  $\nabla^2\phi=0$  inside sphere

$\phi$  should not diverge at origin  $\Rightarrow B_l=0$  for all  $l$

$$\phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Rightarrow \phi(R,\theta) = \phi_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

$$\begin{aligned} \rightarrow \int_0^{\pi} d\theta \sin\theta \phi_0(\theta) P_m(\cos\theta) &= \sum_{l=0}^{\infty} A_l R^l \int_0^{\pi} d\theta \sin\theta P_l(\cos\theta) P_m(\cos\theta) \\ &= \sum_{l=0}^{\infty} A_l R^l \left( \frac{2}{2l+1} \right) \delta_{lm} \end{aligned}$$

$$= A_m R^m \frac{2}{2m+1}$$

$$A_m = \frac{2m+1}{2R^m} \int_0^{\pi} d\theta \sin\theta \phi_0(\theta) P_m(\cos\theta)$$

gives  
solution

To find solution of  $\nabla^2 \phi = 0$  outside sphere

if require  $\phi \rightarrow 0$  as  $r \rightarrow \infty$ , then  $A_l = 0$  for all  $l$

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$\phi(R, \theta) = \phi_0(\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta)$$

gives  
solution

$$B_m = \frac{2m+1}{2} R^{m+1} \int_0^{\pi} \sin \theta \phi_0(\theta) P_m(\cos \theta) d\theta$$

$$B_m = A_m R^{2m+1}$$

- ② Suppose one is given surface charge density  $\sigma(\theta)$  fixed on surface of sphere of radius  $R$ . What is  $\phi$  inside and outside?

From previous example

$$\phi(r, \theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) & r < R \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) & r > R \end{cases}$$

boundary conditions at  $r = R$  on surface

(i)  $\phi$  continuous

$$\rightarrow \sum_{l=0}^{\infty} \left[ A_l R^l - \frac{B_l}{R^{l+1}} \right] P_l(\cos \theta) = 0$$

If an expansion in Legendre polynomials vanishes for all  $\theta$ , then each coefficient in the expansion must vanish

$$\Rightarrow A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow \boxed{B_l = A_l R^{2l+1}}$$

(ii) jump in electric field at  $\sigma$

$$-\left. \frac{\partial \phi^{\text{out}}}{\partial r} \right|_{r=R} + \left. \frac{\partial \phi^{\text{in}}}{\partial r} \right|_{r=R} = 4\pi\sigma$$

$$\Rightarrow \sum_{l=0}^{\infty} \left[ \frac{(l+1)B_l}{R^{l+2}} + lA_l R^{l-1} \right] P_l(\cos\theta) = 4\pi\sigma$$

$$\Rightarrow \sum_{l=0}^{\infty} \left[ \frac{(l+1)A_l R^{2l+1}}{R^{l+2}} + lA_l R^{l-1} \right] P_l(\cos\theta)$$

$$\Rightarrow \sum_{l=0}^{\infty} (2l+1)R^{l-1} A_l P_l(\cos\theta) = 4\pi\sigma$$

$$(2m+1)R^{m-1} A_m \left( \frac{2}{2m+1} \right) = 4\pi \int_0^{\pi} \sin\theta \sigma(\theta) P_m(\cos\theta)$$

$$\boxed{A_m = \frac{4\pi}{2R^{m-1}} \int_0^{\pi} \sin\theta \sigma(\theta) P_m(\cos\theta)}$$



Suppose  $\sigma(\theta) = k \cos \theta$  . what is  $\phi$ ?

Note  $\sigma(\theta) = k P_1(\cos \theta)$

hence only  $A_1 \neq 0$  by orthogonality of  $P_0(\cos \theta)$

$$A_1 = \frac{4\pi k}{2} \int_0^\pi \sin \theta P_1(\cos \theta) P_1(\cos \theta) d\theta$$
$$= \frac{4\pi k}{2} \left( \frac{2}{2+1} \right) = \frac{4\pi k}{3}$$

$$\Rightarrow \phi(r, \theta) = \begin{cases} \frac{4\pi k}{3} r \cos \theta & r < R \\ \frac{4\pi k}{3} \frac{R^3}{r^2} \cos \theta & r > R \end{cases}$$

we will see that potential outside the sphere is that of an ideal dipole with dipole moment

$$p = \frac{4\pi R^3 k}{3}$$

Inside the sphere, the potential  $\phi = \frac{4\pi k}{3} z$  where  $z = r \cos \theta$ . The electric field inside the sphere is therefore the constant

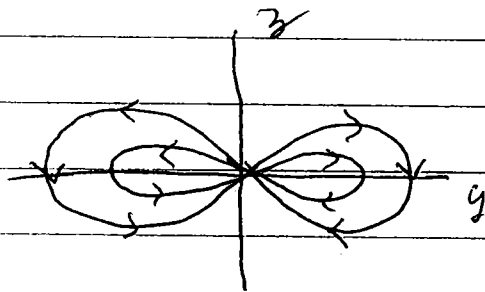
$$\vec{E} = -\vec{\nabla} \phi = -\frac{4\pi k}{3} \hat{z}$$

outside the sphere the field is

$$\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$$

$$= \frac{8\pi k R^3}{3} \frac{\cos\theta}{r^3}\hat{r} + \frac{4\pi k R^3}{3} \frac{\sin\theta}{r^3}\hat{\theta}$$

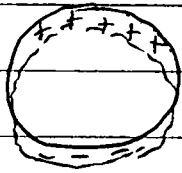
$$\vec{E} = \frac{4\pi R^3 k}{3} \frac{1}{r^3} [2\cos\theta\hat{r} + \sin\theta\hat{\theta}]$$



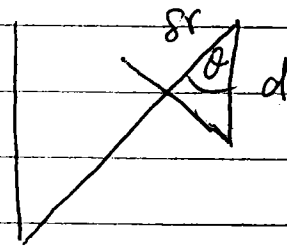
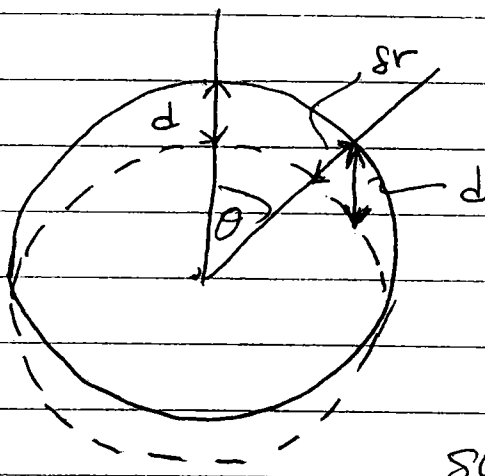
dipole field

Physical example with  $\sigma(\theta) = k \cos \theta$

Two spheres of radii  $R$ , with equal but opposite uniform charge densities  $\rho$  and  $-\rho$ , displaced by small distance  $d \ll R$



Surface charge  $\sigma$  builds up due to displacement  
This is a uniformly "polarized" sphere



$$d \cos \theta = r$$

Surface charge is  $\sigma(\theta) = \rho r$   
 $= \rho d \cos \theta$

$$\sigma(\theta) = \rho d \cos \theta$$

total dipole moment is  $(\rho d) \frac{4}{3} \pi R^3$

polarization =  $\frac{\text{dipole moment}}{\text{volume}} = \rho d$

$\vec{E}$  field inside a uniformly polarized sphere is constant.  $\vec{E} = -\rho d \frac{4\pi}{3}$

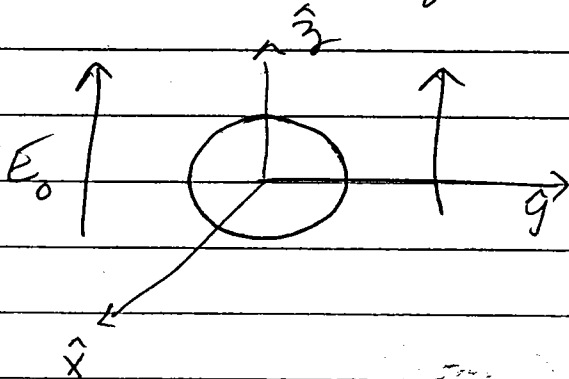
### Grounded

③ Conducting sphere in uniform electric field  $\vec{E} = E_0 \hat{z}$

as  $r \rightarrow \infty$  far from sphere,  $\vec{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z$

boundary conditions  $= -E_0 r \cos \theta$

$$\begin{cases} \phi(R, \theta) = 0 \\ \phi(r \rightarrow \infty, \theta) = -E_0 r \cos \theta \end{cases}$$



∴ solution outside sphere has the form

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos \theta)$$

From boundary condition as  $r \rightarrow \infty$  we have

$$A_l = 0 \quad \text{all } l \neq 1$$

$$A_1 = -E_0 \quad \text{since } P_1(\cos \theta) = \cos \theta$$

$$\phi(r, \theta) = -E_0 r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

From  $\phi(R, \theta) = 0$  we have

$$0 = -E_0 R \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta)$$

$$\Rightarrow B_l = 0 \quad \text{all } l \neq 1$$

$$\frac{B_1}{R^2} = E_0 R \Rightarrow B_1 = +E_0 R^3$$

So  $\phi(r, \theta) = -E_0 \left( r - \frac{R^3}{r^2} \right) \cos \theta$

1<sup>st</sup> term is just potential  $-E_0 r \cos \theta$  of the uniform applied electric field.

2<sup>nd</sup> term is potential due to the induced surface charge on the surface - it is a dipole field

Induced charge density is

$$4\pi\sigma(\theta) = -\left. \frac{\partial\phi}{\partial r} \right|_{r=R} = E_0 \left( 1 + \frac{2R^3}{R^3} \right) \cos \theta$$

$$= 3E_0 \cos \theta$$

$$\sigma(\theta) = \frac{3}{4\pi} E_0 \cos \theta \quad \text{like uniformly polarized sphere} \quad k = \frac{3E_0}{4\pi}$$

from (2) we know that the field inside the sphere due to this  $\sigma$  is just  $-\frac{4}{3}\pi k \hat{z} = -\frac{4}{3}\pi \frac{3E_0}{4\pi} \hat{z}$

$= -E_0 \hat{z}$ . This is just what is required so that the total field in the conducting sphere vanishes,

Can check that outside the sphere,  $\vec{E} = -\vec{\nabla}\phi$  is normal to surface of sphere at  $r=R$ .