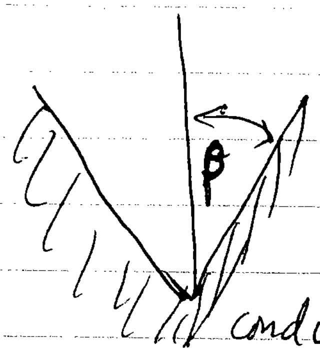
Behavior of fields near conical hole or sharp tip



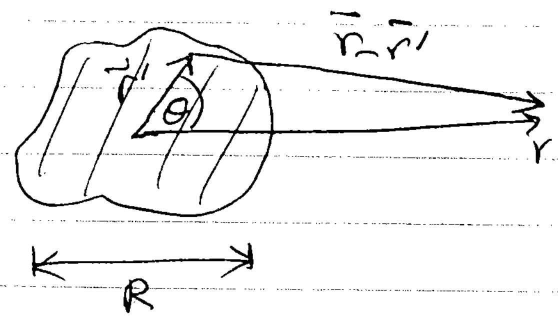
we now want to solve the $\nabla^2\phi = 0$ with separation of variables, but now θ is restricted to range $0 \le \theta \le \beta$.

We still have azimuthal symmetry, but now, since we do not need solution to be finite for all $\theta \in [0, \pi]$, but only $\theta \in (0, \beta)$, we have more solutions to the (14) equation, ie l does not have to be integer, - still need $l \geq 0$ to be finite at $\theta = 0$.

see Jackson sec. 3.4 for details.

Multipole Expansion

region with $\rho \neq 0$

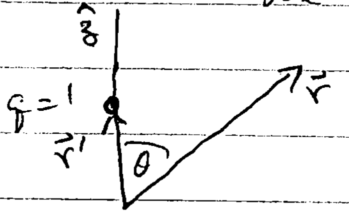


We want to find the potential ϕ for an arbitrary <u>localized</u> distribution of charge ρ, at distances far away $r >> R$.

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \text{General Coulomb formula}$$

We want an expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ in powers of $\left(\frac{r'}{r}\right)$ for $r >> r'$

$\frac{1}{|\vec{r}-\vec{r}'|}$ view this as the potential at $\vec{r}$ due to a unit point charge located at position $\vec{r}'$. We take $\vec{r}'$ on the $\hat{z}$ axis.



The problem has azimuthal symmetry $\Rightarrow \phi$ depends only on r and θ, so we can express it as an expansion in Legendre polynomials.

For $r > r'$,

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

all $A_\ell = 0$ as need $\phi \to 0$ as $r \to \infty$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} P_\ell(\cos\theta)$$

We know $\phi(r, \theta=0) = \dfrac{1}{r-r'}$ (for $r > r'$)

$\leftarrow$ scalars here since when $\theta = 0$, $\vec{r}$ and $\vec{r}\,'$ are both on $\hat{z}$ axis

$$\Rightarrow \phi(r,0) = \frac{1}{r} \sum_{\ell} \frac{B_\ell}{r^\ell} P_\ell(1)$$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} \quad \text{as } P_\ell(1) = 1$$

$$= \frac{1}{r} \frac{1}{(1 - r'/r)} \leftarrow \text{exact result from Coulomb}$$

Now Taylor expansion $\dfrac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots$

$$\Rightarrow \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} = \frac{1}{r}\left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^3 + \cdots\right)$$

$\Rightarrow B_\ell = (r')^\ell$ is solution

So for $r > r'$

$$\boxed{\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)}$$

So for the charge distribution ρ,

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \int d^3r' \frac{\rho(\vec{r}\,')}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3r' \rho(\vec{r}\,') (r')^\ell P_\ell(\cos\theta)$$

where θ is the angle between the fixed observation point $\vec{r}$ and the integration variable $\vec{r}\,'$.

This is the multipole expansion, which expresses the potential far from a localized source as a power series in (r'/r). It is exact provided one adds all the infinite ℓ terms. In practice, one generally approximates by summing only up to some finite ℓ.

<u>Note:</u> in doing the integrals

$$\int d^3r' \, \rho(\vec{r}')(r')^\ell P_\ell(\cos\theta)$$

θ is defined as the angle of $\vec{r}'$ with respect to observation point $\vec{r}$. We therefore in principle have to repeat this integration every time we change $\vec{r}$.

We will find a way around this by

(i) first looking explicitly at the few lowest order terms

(ii) a general method involving spherical harmonics $Y_{\ell m}(\theta, \varphi)$

monopole: $l=0$ term

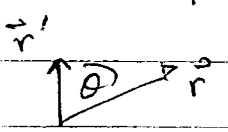
$$\phi^{(0)}(\vec{r}) = \frac{1}{r}\int d^3r'\,\rho(r') \qquad P_0(\cos\theta) = 1$$

$$= \frac{q}{r} \quad \text{where } q \equiv \int d^3r'\,\rho(r') \text{ is}$$

total charge

dipole: $l=1$ term

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2}\int d^3r'\,\rho(\vec{r}\,')\,r'\,P_1(\cos\theta)$$



$$= \frac{1}{r^2}\int d^3r'\,\rho(\vec{r}\,')\,r'\cos\theta$$

Now $\vec{r}\cdot\vec{r}\,' = rr'\cos\theta \Rightarrow \hat{r}\cdot\vec{r}\,' = r'\cos\theta$

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2}\,\hat{r}\cdot\int d^3r'\,\rho(\vec{r}\,')\,\vec{r}\,'$$

$$= \frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{where } \vec{p} \equiv \int d^3r'\,\rho(\vec{r}\,')\,\vec{r}\,'$$

is the *dipole moment*

For a set of point charges q_i at $\vec{r}_i$,

$$\vec{p} = \sum_i q_i\,\vec{r}_i$$

quadrapole : $\ell = 2$ term

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\int d^3r'\, \rho(\vec{r}')\, r'^2\, P_2(\cos\theta)$$

$$= \frac{1}{r^3}\int d^3r'\, \rho(\vec{r}')\, r'^2\, \frac{1}{2}(3\cos^2\theta - 1)$$

use $\cos\theta = \hat{r}'\cdot\hat{r}$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\int d^3r'\, \rho(\vec{r}')\, \frac{1}{2}\left(3\,(\vec{r}'\cdot\hat{r})^2 - (r')^2\right)$$

$$= \frac{1}{r^3}\,\hat{r}\cdot\left[\int d^3r'\rho(\vec{r}')\,\frac{1}{2}\left(3\,\vec{r}'\vec{r}' - (r')^2\overleftrightarrow{I}\right)\right]\cdot\hat{r}$$

where $\overleftrightarrow{I}$ is the identity tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u}\cdot\overleftrightarrow{I}\cdot\vec{v} = \vec{u}\cdot\vec{v}$
and $\vec{r}'\vec{r}'$ is the tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u}\cdot[\vec{r}'\vec{r}']\cdot\vec{v} = (\vec{u}\cdot\vec{r}')(\vec{r}'\cdot\vec{v})$

Define quadrapole tensor $\overleftrightarrow{Q} \equiv \int d^3r'\rho(\vec{r}')\left(3\vec{r}'\vec{r}' - (r')^2\overleftrightarrow{I}\right)$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\,\frac{1}{2}\,\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}$$

So to lowest three terms

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{2r^3} + \cdots$$

defined in terms of the moments q, $\vec{p}$, $\overleftrightarrow{Q}$ of the charge distribution.

Note, the moments q, $\vec{p}$, $\overleftrightarrow{Q}$ do not depend on the observation point $\vec{r}$ – we can calculate them once and then use them to get $\phi(\vec{r})$ at all $\vec{r}$.

monopole: $q = \int d^3r\, \rho(\vec{r}')$ scalar integral

dipole $\vec{p} = \int d^3r\, \rho(\vec{r})\, \vec{r}$ vector integral

$\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, $\hat{e}_3 \equiv \hat{z}$

if we pick a coordinate system, we have to do 3 integrations to get the three component of $\vec{p}$

$$\hat{e}_i \cdot \vec{p} = p_i = \int d^3r\, \rho(\vec{r})\, r_i$$

quadrapole $\overleftrightarrow{Q} = \int d^3r\, \rho(\vec{r})\,(3\vec{r}\,\vec{r} - (r)^2 \overleftrightarrow{I})$ tensor integral

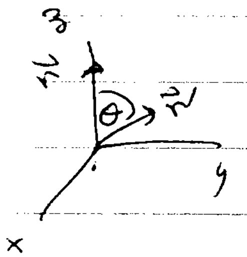
if we pick a coord system $x\ y\ z$ then $\overleftrightarrow{Q}$ is a matrix with components $\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, $\hat{e}_3 \equiv \hat{z}$

$$\hat{e}_i \cdot \overleftrightarrow{Q} \cdot \hat{e}_j = Q_{ij} = \int d^3r\, \rho(\vec{r}) \left[3 r_i r_j - r^2 \delta_{ij} \right]$$

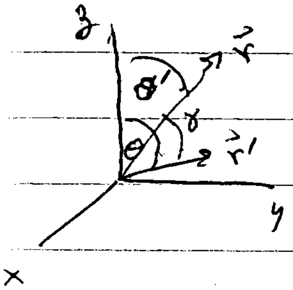
There are 9 elements of the 3×3 matrix Q_{ij}, but $Q_{ij} = Q_{ji}$ is symmetric so there are only 6 independent elements to compute.

General method

$$\phi(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3r' \, \rho(\vec{r}\,')(r')^\ell \, P_\ell(\cos\theta)$$



in above, θ is angle between $\vec{r}$ and $\vec{r}\,'$
if we think of this θ as the spherical coord θ, then in effect, above is choosing $\vec{r}$ to be on $\hat{z}$ axis. We would like a representation in which $\vec{r}$ is positioned arbitrarily with respect to the axes used in describing ρ



Use the <u>addition theorem for spherical harmonics</u>
- see Jackson 3.6 for discussion & proof

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') \, Y_{\ell m}(\theta, \phi)$$

where (θ, ϕ) are the angles of $\hat{r}$, (θ', ϕ') are the angles of $\hat{r}\,'$, and γ is the angle between $\hat{r}$ and $\hat{r}\,'$, ie $\cos\gamma = \hat{r} \cdot \hat{r}\,'$

$\cos\theta = \hat{z} \cdot \hat{r}$

$\cos\theta' = \hat{z} \cdot \hat{r}\,'$

$$\Rightarrow$$

$$\phi(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \int d^3r' \, \rho(\vec{r}\,')(r')^\ell \, Y^*_{\ell m}(\theta', \phi') \, Y_{\ell m}(\theta, \phi)$$

Define the moment

$$q_{\ell m} \equiv \int d^3r' \, \rho(\vec{r}\,')(r')^\ell \, Y^*_{\ell m}(\theta', \phi')$$

independent of observation point

Then

$$\Phi(\vec{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{q_{lm} Y_{lm}(\theta, \phi)}{(2l+1)\, r^{l+1}}$$

see Jackson eqn (4.4), (4.5), (4.6) to relate q_{lm} to q, $\vec{p}$, $\overleftrightarrow{Q}$.

$$\Phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{2r^3} \cdots$$

electric field $\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi}$

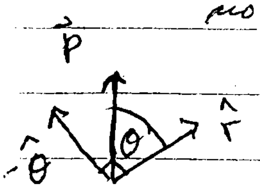
For the monopole term $\vec{E} = \frac{q}{r^2}\hat{r}$

For the dipole term, choose $\vec{p}$ along $\hat{z}$ axis so

$$\Phi(\vec{r}) = \frac{p\cos\theta}{r^2}$$

$$\vec{E} = \frac{2p\cos\theta}{r^3}\hat{r} + \frac{p\sin\theta}{r^3}\hat{\theta}$$

$$\vec{E} = \frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



note $p\cos\theta\,\hat{r} = (\vec{p}\cdot\hat{r})\hat{r}$

$p\sin\theta\,\hat{\theta} = -(\vec{p}\cdot\hat{\theta})\hat{\theta}$

Now $\vec{p} = (\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{\theta})\hat{\theta}$

$\Rightarrow -(\vec{p}\cdot\hat{\theta})\hat{\theta} = (\vec{p}\cdot\hat{r})\hat{r} - \vec{p}$

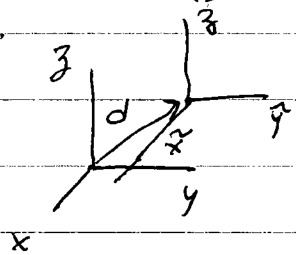
So

$$\vec{E} = \frac{1}{r^3}\left[2(\vec{p}\cdot\hat{r})\hat{r} + (p\cdot\hat{r})\hat{r} - \vec{p}\right]$$

$$\vec{E} = \frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p} \right]$$ expresses $\vec{E}$ of dipole in coord free form

Origin of coordinates

The definition of the multipole moments depends on the choice of origin of the coordinates.



Suppose transform to $\tilde{\vec{r}} = \vec{r} - \vec{d}$

In the $\tilde{\vec{r}}$ cood system

$$\tilde{q} = \int d^3\tilde{r}\, \rho(\tilde{r}) = \int d^3r\, \rho(r) = q$$

monopole does not depend on choice of origin

$$\tilde{\vec{p}} = \int d^3\tilde{r}\, \rho(\tilde{r})\, \tilde{\vec{r}} = \int d^3r\, \rho\,(\vec{r}-\vec{d})$$

$$= \int d^3r\, \rho\, \vec{r} - \vec{d}\int d^3r\, \rho$$

$$\tilde{\vec{p}} = \vec{p} - \vec{d}q \qquad \tilde{\vec{p}} = \vec{p} \text{ only if } q = 0\,!$$

if $q \neq 0$, then $\tilde{\vec{p}} \neq \vec{p}$

$\Rightarrow$ ~~One coord~~ If $q \neq 0$, one could always choose ~~an~~ origin of coords for which $\vec{p} = 0$!

For HW you will show that $\tilde{\overleftrightarrow{Q}} = \overleftrightarrow{Q}$ only if both $q = 0$ and $\vec{p} = 0$.

Example two charges q_1 at $\vec{r}_1$ and q_2 at $\vec{r}_2$

monopole $q = q_1 + q_2$

dipole $\vec{p} = q_1 \vec{r}_1 + q_2 \vec{r}_2$

quadrupole $Q_{ij} = (3 r_{1i} r_{1j} - r_1^2 \delta_{ij}) q_1 + (3 r_{2i} r_{2j} - r_2^2 \delta_{ij}) q_2$

Dipole

① Suppose $q_1 = -q_2$, so $q = 0$

then $\vec{p} = q_1 (\vec{r}_1 - \vec{r}_2)$ independent of choice of origin

② Suppose $q_1 + q_2 \neq 0$

then $\vec{p} = q_1 \vec{r}_1 + q_2 \vec{r}_2$

choose $\vec{d} \equiv \frac{\vec{p}}{q}$ Define new coords $\vec{r}\,' = \vec{r} - \vec{d}$

then in $\vec{r}\,'$ coord system

$$\vec{p}\,' = q_1 \vec{r}_1\,' + q_2 \vec{r}_2\,' = q_1 (\vec{r}_1 - \vec{d}) + q_2 (\vec{r}_2 - \vec{d})$$
$$= \vec{p} - q\vec{d} = \vec{p} - q\left(\frac{\vec{p}}{q}\right) = 0$$

$\vec{p}\,'$ dipole moment vanishes! The location of the charges in the $\vec{r}\,'$ coord system is given by:

$$\vec{r}_1\,' = \vec{r}_1 - \frac{\vec{p}}{q} = \vec{r}_1 - \frac{(q_1 \vec{r}_1 + q_2 \vec{r}_2)}{q_1 + q_2} = \frac{q_2 (\vec{r}_1 - \vec{r}_2)}{q_1 + q_2}$$

$$\vec{r}_2\,' = \vec{r}_2 - \frac{\vec{p}}{q} = \vec{r}_2 - \frac{(q_1 \vec{r}_1 + q_2 \vec{r}_2)}{q_1 + q_2} = \frac{q_1 (\vec{r}_2 - \vec{r}_1)}{q_1 + q_2}$$

$\vec{r}' = 0$ is "center of charge" (like center of mass in mechanics)

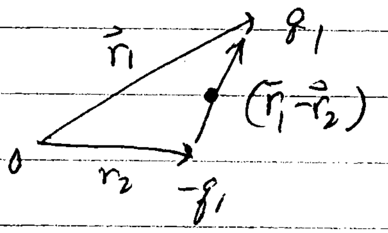
Quadrapole

① Suppose $q_1 + q_2 = q = 0 \Rightarrow q_1 = -q_2$

$$\vec{p} = q_1(\vec{r}_1 - \vec{r}_2) \quad \text{indep of origin}$$

$$Q_{ij} = \left[(3r_{1i} r_{1j} - r_1^2 \delta_{ij}) - (3 r_{2i} r_{2j} - r_2^2 \delta_{ij}) \right] q_1$$

If we choose origin midway between the two charges



$$\vec{r}_1{}' = \vec{r}_1 - \frac{(\vec{r}_1 + \vec{r}_2)}{2} = \frac{\vec{r}_1 - \vec{r}_2}{2}$$

$$\vec{r}_2{}' = r_2 - \frac{(\vec{r}_1 + \vec{r}_2)}{2} = \frac{\vec{r}_2 - \vec{r}_1}{2} = -\vec{r}_1{}'$$

Then with respect to this origin

$$Q'_{ij} = q_1 \left[(3\, r'_{1i} r'_{1j} - 3(r_1')^2 \delta_{ij}) - (3(-r'_{1i})(-r'_{1j}) - 3(-r_1')^2 \delta_{ij}) \right]$$

$$= 0$$

So for this single case, ~~the qu~~ we can make the quadrapole moment vanish

(c) Suppose $q_1 + q_2 = q \neq 0$

Choose origin so that $\vec{p}' = 0$ $\Rightarrow \begin{cases} \vec{r}_1{}' = \dfrac{q_2(\vec{r}_1 - \vec{r}_2)}{q_1 + q_2} \\ \vec{r}_2{}' = \dfrac{q_1(\vec{r}_2 - \vec{r}_1)}{q_1 + q_2} \end{cases}$

Define $\delta\vec{r} = \vec{r}_1 - \vec{r}_2$

$$Q'_{ij} = (3r'_{1i} r'_{1j} - r_1'^2 \delta_{ij}) q_1 + (3r'_{2i} r'_{2j} - r_2'^2 \delta_{ij}) q_2$$

$$= 3(\delta r_i \delta r_j - \delta r^2 \delta_{ij}) \frac{q_1 q_2^2}{(q_1+q_2)^2} + 3((-\delta r_i)(-\delta r_j) - (-\delta r)^2 \delta_{ij}) \frac{q_2 q_1^2}{(q_1+q_2)^2}$$

$$= [3 \delta r_i \delta r_j - \delta r^2 \delta_{ij}] \left[\frac{q_1 q_2^2 + q_2 q_1^2}{(q_1+q_2)^2} \right]$$

$$= (3 \delta r_i \delta r_j - \delta r^2 \delta_{ij}] \frac{q_1 q_2}{q_1 + q_2}$$

Q'_{ij}

Choose $\hat{z}$ axis aligned with $\delta\vec{r}$, ie $\delta\vec{r} = s\hat{z}$

s is distance between the two charges

$$Q'_{ij} = \begin{pmatrix} -s^2 & 0 & 0 \\ 0 & -s^2 & 0 \\ 0 & 0 & 2s^2 \end{pmatrix} \frac{q_1 q_2}{q_1 + q_2}$$

In spherical coords $\hat{r} = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$

$$\Rightarrow \overleftrightarrow{Q}\cdot\hat{r} = \frac{q_1 q_2}{q_1+q_2}\begin{pmatrix} -s^2 & 0 & 0 \\ 0 & -s^2 & 0 \\ 0 & 0 & 2s^2 \end{pmatrix}\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$= \frac{q_1 q_2}{q_1+q_2}\begin{pmatrix} -s^2\sin\theta\cos\varphi \\ -s^2\sin\theta\sin\varphi \\ 2s^2\cos\theta \end{pmatrix}$$

$$\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r} = \frac{q_1 q_2}{q_1+q_2}\left(-s^2\sin^2\theta\cos^2\varphi - s^2\sin^2\theta\sin^2\varphi + 2s^2\cos^2\theta\right)$$

$$= \frac{q_1 q_2}{q_1+q_2}\, s^2\left(2\cos^2\theta - \sin^2\theta\right)$$

$$\Phi = \frac{q}{r} + \frac{1}{2}\frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{r^3} \qquad (p=0)$$

$$\Phi(\vec{r}) = \frac{q}{r} + \frac{s^2}{r^3}\left(\frac{q_1 q_2}{q_1+q_2}\right)\left(2\cos^2\theta - \sin^2\theta\right)$$

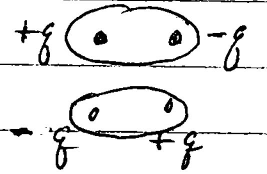
Example

sample charge config's

$\circ\ q$ $\Rightarrow$ monopole is leading term

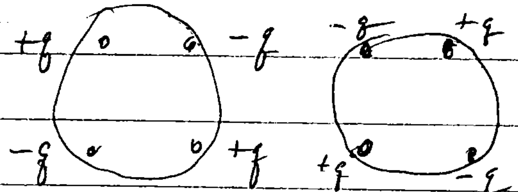
$\underset{+q}{\circ}\ \ \underset{-q}{\circ}$ $\Rightarrow$ monopole $= 0$ $\Rightarrow$ dipole is leading term

$\vec{p}$ is indep of origin



$\Rightarrow$ monopole $= 0$ $\Rightarrow$ total dipole is sum of dipoles of individual neutral pairs

$= 0$ leading term is quadrapole



when monopole $= 0$ and dipole $= 0$, quadrapole is indep of origin.

$\rightarrow$ total quadrapole is sum of quadrapoles of individual clusters with $q = 0$ and $\vec{p} = 0$

$$Q = Q_1 + Q_2$$

with $Q_2 = -Q_1$

$\Rightarrow Q = 0$ leading term is <u>octopole</u>