1) Lorentz Gauge

gauge constraint: require $\frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

Then Gauss' Law becomes

$$\nabla^2 \phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right) = -4\pi\rho$$

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho}$$

Ampere's Law becomes

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}_{=0}$$

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}}$$

The combination $-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \equiv \Box^2$ is the wave equation operator.

In Lorentz gauge, $\vec{A}$ and ϕ satisfy the inhomogeneous wave equations:

$$\boxed{\begin{aligned}\Box^2\vec{A} &= \frac{4\pi}{c}\vec{J} \\ \Box^2\phi &= 4\pi\rho\end{aligned}}$$

when $\vec{J}=0, \rho=0$ electromagnetic waves are solution!

Note: Lorentz gauge condition does not uniquely determine $\vec{A}$ and ϕ. If one ~~constructs~~ has $\vec{A}$ and ϕ obeying Lorentz gauge condition, and then constructs

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$
$$\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$$

Then $\vec{A}'$ and ϕ' will also be in Lorentz gauge provided $\Box^2 \chi = 0$ (proof left to reader)

2) Coulomb Gauge

gauge constraint: require $\vec{\nabla}\cdot\vec{A} = 0$
if $\vec{A}$ is in the Coulomb Gauge, then $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ will also be in Coulomb gauge provided $\nabla^2 \chi = 0$.

Then Gauss' law becomes

$$\nabla^2 \phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \boxed{\nabla^2 \phi = -4\pi\rho} \quad \text{same as electrostatics!}$$

$$\Rightarrow \phi(\vec{r}, t) = \int d^3r' \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|}$$

no matter what motion the source $\rho(\vec{r}, t)$ has! ϕ is given by the instantaneous Coulomb potential even though electromagnetic fields have a finite velocity of propagation c!

Ampere's Law becomes:

$$-\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)$$

$$\Rightarrow \Box^2 A = \frac{4\pi}{c}\vec{J} - \frac{1}{c}\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right)$$

where $\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = \vec{\nabla}\left[\int d^3r' \frac{\partial\rho}{\partial t}\frac{1}{|\vec{r}-\vec{r}'|}\right]$

$$= -\vec{\nabla}\left[\int d^3r' \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\right] \quad \text{by continuity eqn.}$$

To see the meaning of this term, recall - any vector function $\vec{J}$ can be written as the sum of a curlfree and a divergenceless part

$$\vec{J} = \vec{J}_{\parallel} + \vec{J}_{\perp} \quad \text{where} \quad \vec{\nabla}\times\vec{J}_{\parallel} = 0 \quad \text{curlfree}$$
$$\vec{\nabla}\cdot\vec{J}_{\perp} = 0 \quad \text{divergenceless}$$

where

$$\vec{J}_{\parallel}(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla}\int d^3r' \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad \text{longitudinal part}$$

$$\vec{J}_{\perp}(\vec{r}) = \quad \text{transverse part}$$

$$= \frac{1}{4\pi}\vec{\nabla}\times\int d^3r' \frac{\vec{\nabla}'\times\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

So $\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = 4\pi\vec{J}_{\parallel}$, and

$$\Box^2 A = \frac{4\pi}{c}\vec{J} - \frac{4\pi}{c}\vec{J}_{\parallel} = \frac{4\pi}{c}\vec{J}_{\perp}$$

Transverse + Longitudinal Parts of vector functions

To prove the preceeding claim, $\vec{j} = \vec{j}_\parallel + \vec{j}_\perp$, where $\vec{\nabla}\times\vec{j}_\parallel = 0$ and $\vec{\nabla}\cdot\vec{j}_\perp = 0$, we first digress to prove Helmholtz Theorem.

<u>Helmholtz Theorem</u>: For a vector function $\vec{f}(\vec{r})$ if one knows the divergence and curl of $\vec{f}$ then one can ~~uniquely~~ uniquely determine $\vec{f}$ itself.

That is, if

$$\vec{\nabla}\cdot\vec{f} = 4\pi D(\vec{r})$$ where $D(\vec{r})$ is a known scalar function

$$\vec{\nabla}\times\vec{f} = 4\pi\vec{C}(\vec{r})$$ where $\vec{C}(\vec{r})$ is a known vector function.

~~Then one can solve for~~

And if well defined <u>boundary conditions</u> on $\vec{f}$ are known (here we will assume $\vec{f}(\vec{r})\to 0$ as $\vec{r}\to\infty$) Then there is a unique solution for $\vec{f}(\vec{r})$.

We prove this by construction!

<u>Assume</u> a solution of the form

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$ where φ is a scalar and $\vec{W}$ a vector

Now we show that we can find such a solution

First consider

$$\vec{\nabla}\cdot\vec{f} = -\nabla^2\varphi + \vec{\nabla}\cdot(\vec{\nabla}\times\vec{W}) = -\nabla^2\varphi + 0 = 4\pi D(\vec{r})$$

So $\quad -\nabla^2\varphi = 4\pi D(\vec{r})$ This is just Poisson's equ we saw in electrostatics

Solution when $\varphi(\vec{r})\to 0$ as $\vec{r}\to\infty$ is given by

$$\boxed{\varphi(\vec{r}) = \int d^3r' \, \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

Coulomb-like integral solution

Now consider

$$\vec{\nabla}\times\vec{f} = -\vec{\nabla}\times\vec{\nabla}\varphi + \vec{\nabla}\times(\vec{\nabla}\times\vec{W}) = 0 - \nabla^2\vec{W} + \vec{\nabla}(\vec{\nabla}\cdot\vec{W})$$
$$= 4\pi\vec{C}(\vec{r})$$

Choose a gauge in which $\vec{\nabla}\cdot\vec{W} = 0$ (just like Coulomb gauge in magnetostatics)

Then $\quad -\nabla^2\vec{W} = 4\pi\vec{C}(\vec{r})$

$$\boxed{\vec{W}(\vec{r}) = \int d^3r' \, \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

just like solution for vector pot $\vec{A}$ in magnetostatics

So we have constructed a solution

$$f(\vec{r}) = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$

$$= -\vec{\nabla}\int d^3r' \, \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \vec{\nabla}\times\int d^3r' \, \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

where $\vec{\nabla}\cdot\vec{f} = 4\pi D$ and $\vec{\nabla}\times\vec{f} = 4\pi\vec{C}$

Note: For above solution to be well defined, the integrals must converge. They will converge if the "sources" $D(\vec{r})$ and $\vec{C}(\vec{r})$ are sufficiently "localized" in space, ie $D(\vec{r}) \to 0$, $\vec{C}(\vec{r}) \to 0$ sufficiently fast as $\vec{r} \to \infty$.

Now we show that the above solution is unique.

Suppose there was another solution $\vec{g}$ such that

$$\vec{\nabla} \cdot \vec{g} = 4\pi D \quad \text{and} \quad \vec{\nabla} \times \vec{g} = 4\pi \vec{C}$$

Consider $\vec{h} \equiv \vec{f} - \vec{g}$ then

$$\vec{\nabla} \cdot \vec{h} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{h} = 0$$

Can show that only such $\vec{h}$ that also has $\vec{h}(\vec{r}) \to 0$ as $\vec{r} \to \infty$ is $\vec{h} \equiv 0$, so $\vec{g} = \vec{f}$ and solution is unique.

As a consequence of Helmholtz Theorem we have also shown the following

① Any vector function $\vec{f}$ can be written in terms of a scalar and vector potential

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla} \times \vec{W}$$

or equivalently

(2) Any vector function $\vec{f}$ can be written in terms of a curl free and a divergenceless part

$$\vec{f} = \vec{f}_{\parallel} + \vec{f}_{\perp} \qquad \text{where} \quad \vec{\nabla}\times\vec{f}_{\parallel} = 0 \quad \text{curlfree}$$
$$\vec{\nabla}\cdot\vec{f}_{\perp} = 0 \quad \text{divergenceless}$$

where
$$\begin{cases} \vec{f}_{\parallel}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}) = -\vec{\nabla}\int \frac{d^3r'}{4\pi} \frac{[\vec{\nabla}'\cdot\vec{f}(\vec{r}')]}{|\vec{r}-\vec{r}'|} \\ \vec{f}_{\perp}(\vec{r}) = \vec{\nabla}\times\vec{W}(\vec{r}) = \vec{\nabla}\times\int \frac{d^3r'}{4\pi} \frac{[\vec{\nabla}'\times\vec{f}(\vec{r}')]}{|\vec{r}-\vec{r}'|} \end{cases}$$

where in above we used $\vec{D}(\vec{r}') = \frac{1}{4\pi}\vec{\nabla}'\cdot\vec{f}(\vec{r}')$

$$\vec{C}(\vec{r}') = \frac{1}{4\pi}\vec{\nabla}'\times\vec{f}(\vec{r}')$$

~~where~~ $\vec{f}_{\parallel}$ is called the longitudinal part of $\vec{f}$

$\vec{f}_{\perp}$ is called the transverse part of $\vec{f}$

To understand the reason for these names, we need to consider the Fourier transforms

Above can be generalized to situations where $\vec{f}$ satisfies other boundary conditions, say has a specified value on a given boundary surface. One first replaces $\frac{1}{|\vec{r}-\vec{r}'|}$ by the appropriate Greens function — see more to come!

Discussion regarding Fourier transforms

$$\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \vec{f}(\vec{k}) \qquad \text{Fourier transf}$$

$$\vec{f}(\vec{k}) = \int_{-\infty}^{\infty} d^3r \, e^{-i\vec{k}\cdot\vec{r}} \vec{f}(\vec{r}) \qquad \text{inverse transf}$$

Some special cases well worth remembering

① Transform of Dirac function

$$\delta_{\vec{r}_0}(\vec{k}) \equiv \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \delta(\vec{r}-\vec{r}_0) = e^{-i\vec{k}\cdot\vec{r}_0}$$

$$\Rightarrow \quad \delta(\vec{r}-\vec{r}_0) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \delta_{\vec{r}_0}(\vec{k})$$

$$\boxed{\delta(\vec{r}-\vec{r}_0) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}}$$

or letting $\vec{r} \leftrightarrow \vec{k}$ in the above

$$\delta(\vec{k}-\vec{k}_0) = \int \frac{d^3r}{(2\pi)^3} e^{i\vec{r}\cdot(\vec{k}-\vec{k}_0)}$$

② Transform of Coulomb potential $\frac{1}{|\vec{r}-\vec{r}'|}$

We know

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

Suppose $f(\vec{k}) \equiv \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|}$ is the

Fourier transf of $\frac{1}{|\vec{r}-\vec{r}'|}$

Substitute
$$\begin{cases} \dfrac{1}{|r-r'|} = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} f(\vec{k}) \\ \delta(\vec{r}-\vec{r}') = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \end{cases}$$

into above Poisson equation

$$\underbrace{\nabla^2 \int \frac{d^3k}{(2\pi)^3}}\, e^{i\vec{k}\cdot\vec{r}} f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

operates only on $\vec{r}$
so move inside integral

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right)$$

① $$\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i \frac{\partial}{\partial x_i} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i\, i k_i\, e^{i\vec{k}\cdot\vec{r}}$$

$$= i\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \qquad \text{where } \hat{x}_1, \hat{x}_2, \hat{x}_3 = \hat{x}, \hat{y}, \hat{z}$$

② $$\vec{\nabla}\cdot\left(i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\right) = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

so $$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

Poisson eqn then gives

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} (-k^2) f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}\cdot\vec{r}'}$$

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-k^2 f(\vec{k})\right] = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-4\pi e^{-i\vec{k}\cdot\vec{r}'}\right]$$

As is true for Fourier series, so it is true for Fourier transforms: If two functions are equal, then their Fourier transforms are equal.

$$\Rightarrow \quad -k^2 f(\vec{k}) = -4\pi e^{-i\vec{k}\cdot\vec{r}'}$$

$$\boxed{f(\vec{k}) = \frac{4\pi}{k^2} e^{-i\vec{k}\cdot\vec{r}'}}$$

$\Rightarrow$ is the Fourier transform of $\frac{1}{|\vec{r}-\vec{r}'|}$

Now to see the meaning of the decomposition

$$\vec{f}(\vec{r}) = \vec{f}_{\parallel}(\vec{r}) + \vec{f}_{\perp}(\vec{r})$$

① one way to do it:

$$\vec{f}_{\parallel}(\vec{r}) = -\vec{\nabla}\varphi \qquad \text{where we had } -\nabla^2\varphi = 4\pi D$$

$$\vec{f}_{\perp}(\vec{r}) = \vec{\nabla}\times\vec{W} \qquad \text{where we had } -\nabla^2\vec{W} = 4\pi\vec{C}$$

where $D = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{f}$

and $\vec{C} = \frac{1}{4\pi}\vec{\nabla}\times\vec{f}$

$$\Rightarrow -\nabla^2\varphi = \vec{\nabla}\cdot\vec{f}$$

$$-\nabla^2\vec{W} = \vec{\nabla}\times\vec{f}$$

write in terms of the Fourier transforms of φ, $\vec{W}$, $\vec{f}$, the above equations become

$$-\nabla^2 \underbrace{\int \frac{d^3k}{(2\pi)^3}}_{\text{move inside}} e^{i\vec{k}\cdot\vec{r}} \varphi(\vec{k}) = \vec{\nabla}\cdot \underbrace{\int \frac{d^3k}{(2\pi)^3}}_{\text{move inside}} e^{i\vec{k}\cdot\vec{r}} \vec{f}(\vec{k})$$

$$\Rightarrow \int \frac{d^3k}{(2\pi)^3} \left(-\nabla^2 e^{i\vec{k}\cdot\vec{r}}\right) \varphi(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} \left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right)\cdot \vec{f}(k)$$

$$\Rightarrow \int \frac{d^3k}{(2\pi)^3}\, k^2 e^{i\vec{k}\cdot\vec{r}} \varphi(\vec{k}) = \int \frac{d^3k}{(2\pi)^3}\, i\vec{k}\cdot e^{i\vec{k}\cdot\vec{r}} \vec{f}(\vec{k})$$

equate Fourier transforms

$$\Rightarrow \boxed{k^2 \varphi(k) = i\vec{k}\cdot\vec{f}(k)}$$

Similarly, $-\nabla^2 \vec{W} = \vec{\nabla}\times\vec{f}$ gives

$$\boxed{k^2 \vec{W}(\vec{k}) = i\vec{k}\times f(\vec{k})}$$

Now insert Fourier transforms into

$$\vec{f}_{\parallel} = -\nabla\varphi$$
$$\vec{f}_{\perp} = \vec{\nabla}\times\vec{W}$$

to get

$$\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \vec{f}_{\parallel}(k) = -\vec{\nabla} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \varphi(\vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3} (-i\vec{k})\, \varphi(\vec{k})$$

equate Fourier ~~coefficie~~ transforms

$$\boxed{\vec{f}_{\parallel}(\vec{k}) = -i\vec{k}\,\varphi(\vec{k})}$$

Similarly we get

$$\boxed{\vec{f}_\perp(\vec{k}) = i\vec{k} \times \vec{W}(\vec{k})}$$

Combine results to get

$$\vec{f}_{\parallel}(\vec{k}) = -i\vec{k}\,\varphi(\vec{k}) = (-i\vec{k})\left(\frac{i\,\vec{k}\cdot\vec{f}(\vec{k})}{k^2}\right)$$

$$\boxed{\vec{f}_{\parallel}(\vec{k}) = \hat{k}\left(\hat{k}\cdot\vec{f}(\vec{k})\right)} \qquad \text{where } \hat{k} = \frac{\vec{k}}{k}$$

$$\vec{f}_\perp(\vec{k}) = i\vec{k}\times\vec{W}(\vec{k}) = i\vec{k}\times\left(\frac{i\,\vec{k}\times\vec{f}(\vec{k})}{k^2}\right)$$

$$\boxed{\vec{f}_\perp(\vec{k}) = -\hat{k}\times\left(\hat{k}\times\vec{f}(\vec{k})\right)}$$

we can use triple product rule to rewrite above as

$$-\hat{k}\times\left(\hat{k}\times\vec{f}(\vec{k})\right) = -\hat{k}\left(\hat{k}\cdot\vec{f}(\vec{k})\right) + \vec{f}(\vec{k})\left(\hat{k}\cdot\hat{k}\right)$$

$$= \vec{f}(\vec{k}) - \hat{k}\cdot\left(\hat{k}\cdot\vec{f}(\vec{k})\right)$$

$$= \vec{f}(\vec{k}) - \vec{f}_{\parallel}(\vec{k})$$

$\Rightarrow$ in terms of Fourier transforms, $\vec{f}_{\parallel}(\vec{k})$ is component of $\vec{f}(\vec{k})$ that is parallel to wavevector $\vec{k}$, while $\vec{f}_\perp(\vec{k})$ is the component perpendicular to $\vec{k}$. Hence they are called the longitudinal & transverse parts

(2) Another direct way

$$\vec{f}_{\parallel}(\vec{r}) = -\vec{\nabla}\int \frac{d^3r'}{4\pi}\,\frac{\vec{\nabla}'\cdot \vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

substitute into the above the Fourier transforms

$$\frac{1}{|\vec{r}-\vec{r}'|} = \int \frac{d^3k'}{(2\pi)^3}\, e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')}\,\frac{4\pi}{(k')^2}$$

$$\vec{f}(\vec{r}') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}'}\,\vec{f}(\vec{k})$$

$$\Rightarrow \vec{f}_{\parallel}(\vec{r}) = -\vec{\nabla}\int \frac{d^3r'}{4\pi}\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\, e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')}\,\frac{4\pi}{(k')^2}$$

$$\times\ \vec{\nabla}'\cdot\left[e^{i\vec{k}\cdot\vec{r}'}\,\vec{f}(\vec{k})\right]$$

we need $\vec{\nabla} e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')} = i\vec{k}'\, e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')}$

acts on $\vec{r}$

and $\vec{\nabla}'\cdot\left[e^{i\vec{k}\cdot\vec{r}'}\vec{f}(\vec{k})\right] = \left[\vec{\nabla}' e^{i\vec{k}\cdot\vec{r}'}\right]\cdot\vec{f}(\vec{k})$

$= i\vec{k}\, e^{i\vec{k}\cdot\vec{r}'}\cdot\vec{f}(\vec{k})$

$$\Rightarrow \vec{f}_{\parallel}(\vec{r}) = -\int \frac{d^3r'}{4\pi}\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\, e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')}\,(i\vec{k}')\,\frac{4\pi}{(k')^2}$$

$$\times\ i\vec{k}\cdot\vec{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}'}$$

group the terms in $\vec{r}'$ together and do $\int d^3r'$ first

Returning to Ampere's law we see that the term

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right) = -\vec{\nabla} \int d^3r' \left[\frac{\vec{\nabla}' \cdot \vec{j}(r', t)}{|\vec{r}-\vec{r}'|}\right]$$

$$= 4\pi \vec{j}_{\parallel}(\vec{r}, t)$$

So Amperes law becomes

$$\Box^2 \vec{A} = \frac{4\pi}{c}\vec{j} - \frac{4\pi}{c}\vec{j}_{\parallel}$$

$$\boxed{\Box^2 \vec{A} = \frac{4\pi}{c}\vec{j}_{\perp}}$$

In Coulomb gauge, only the transverse part of $\vec{j}$ serves as a source for $\vec{A}$.

$\vec{A}$ describes the transverse modes, ie the EM radiation (recall in EM waves, the fields are always $\perp$ direction of propagation)

ϕ describes the longitudinal modes

Coulomb gauge is not Lorentz invariant - if $\vec{\nabla} \cdot \vec{A} = 0$ in one inertial reference frame, in general $\vec{\nabla} \cdot \vec{A} \neq 0$ in another.

In Coulomb gauge, if $\rho = 0$, then $\phi = 0$ and

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$