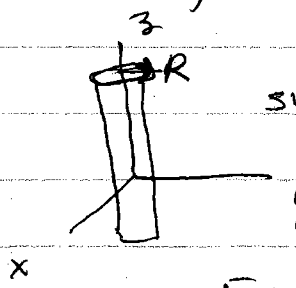
Some more problems

infinite conducting wire of radius R with line charge density λ = charge per unit length



surface charge $\sigma = \frac{\lambda}{2\pi R}$

Expect cylindrical symmetry $\Rightarrow \phi$ depends only on cylindrical coord r.

$\nabla^2\phi = 0$ for $r > R$, $r < R$

use ∇^2 in cylindrical coords – only radial term non vanishing

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

$$r\frac{d\phi}{dr} = C_0 \quad \text{constant}$$

$$\frac{d\phi}{dr} = \frac{C_0}{r}$$

$$\phi(r) = C_0 \ln r + C_1 \quad \text{const}$$

note: one cannot now choose $\phi \to 0$ as $r \to \infty$!

one needs to fix zero of ϕ at some other radius. a convenient choice is $r = R$, but any other choice could also be made.

$$\phi^{out} = C_0^{out} \ln r + C_1^{out}$$
$$\phi^{in} = C_0^{in} \ln r + C_1^{in}$$

ϕ^{in} = const in conductor $\rightarrow C_0^{in} = 0$
or ϕ^{in} should not diverge as $r \to 0 \Rightarrow C_0^{in} = 0$

so $\phi^{in} = C_1^{in}$ constant

boundary condition at $r = R$

$$\left[-\frac{d\phi^{out}}{dr} + \frac{d\phi^{in}}{dr} \right]_{r=R} = 4\pi\sigma$$

$$\Rightarrow \quad -\frac{C_0^{out}}{R} = 4\pi\sigma = 4\pi\left(\frac{\lambda}{2\pi R}\right) = \frac{2\lambda}{R}$$

$$C_0^{out} = -2\lambda$$

$$\phi^{out}(r) = -2\lambda \ln r + C_1^{out}$$

continuity of ϕ

$$\phi^{in}(R) = \phi^{out}(R) \quad \Rightarrow \quad C_1^{in} = -2\lambda \ln R + C_1^{out}$$

Remaining const C_1^{out} is not too important as it is just a common additive constant to both ϕ^{in} and ϕ^{out} $\Rightarrow$ does not change $\vec{E} = -\vec{\nabla}\phi$.

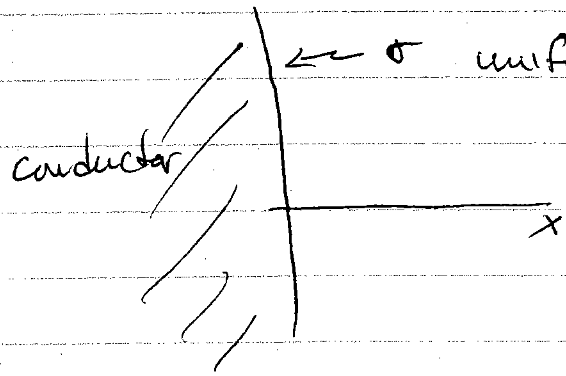
If use the condition $\phi(R) = 0$ then we can solve for C_1^{out}.

$$0 = -2\lambda \ln R + C_1^{out} \quad \Rightarrow \quad C_1^{out} = 2\lambda \ln R$$

$$\Rightarrow \quad \phi(r) = \begin{cases} -2\lambda \ln(r/R) & r > R \\ 0 & r < R \end{cases}$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \begin{cases} \frac{2\lambda}{r}\,\hat{r} & r > R \\ 0 & r < R \end{cases}$$

infinite conducting half space



uniform surface charge density

expect ϕ depends only on x

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = 0$$

$$\rightarrow \begin{cases} \phi^>(x) = c_0^> x + c_1^> & x > 0 \\ \phi^<(x) = c_0^< x + c_1^< & x < 0 \end{cases}$$

for $x < 0$, $\phi = \text{const}$ in conductor $\Rightarrow c_0^< = 0$

at $x = 0$, ϕ continuous $\Rightarrow \phi^<(0) = \phi^>(0)$

$$c_1^< = c_1^>$$

$\frac{d\phi}{dx}$ discontinuous $\Rightarrow$

$$-\left.\frac{d\phi^>}{dx}\right|_{x=0} = 4\pi\sigma$$

$$c_0^> = -4\pi\sigma$$

$$\Rightarrow \quad \phi(x) = \begin{cases} -4\pi\sigma x + c_1^> & x > 0 \\ c_1^> & x < 0 \end{cases}$$

const $c_1^>$ does not change value of $\vec{E}$

as for the wire, we cannot choose $\phi \to 0$ as $x \to \infty$.
~~we can set $\phi = 0$ at~~

$$-\vec{\nabla}\phi = \vec{E} = \begin{cases} 4\pi\sigma\,\hat{x} & x > 0 \\ 0 & x < 0 \end{cases}$$

infinite charged plane

similar to previous problem, but now no conductor at $x < 0$, just free space on both sides of the charged plane at $x = 0$.

~~expect ϕ depends only on $|x|$ by symmetry~~

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = 0 \quad \Rightarrow \quad \begin{aligned} \phi^> &= c_0^> x + c_1^> & x > 0 \\ \phi^< &= c_0^< x + c_1^< & x < 0 \end{aligned}$$

continuity of ϕ at $x = 0$

$$\to \phi^>(0) = \phi^<(0) \quad \Rightarrow \quad c_1^> = c_1^<$$

discontinuity of $d\phi/dx$ at $x = 0$

$$-\frac{d\phi^>}{dx} + \frac{d\phi^<}{dx} = 4\pi\sigma$$

$$-c_0^> + c_0^< = 4\pi\sigma$$

Define $\bar{c}_0 = \dfrac{c_0^> + c_0^<}{2}$

Then we can write

$$C_0^< = \bar{C}_0 + 2\pi\sigma$$
$$C_0^> = \bar{C}_0 - 2\pi\sigma$$

$$\phi = \begin{cases} -2\pi\sigma x + \bar{C}_0 x + C_1^> & x > 0 \\ 2\pi\sigma x + \bar{C}_0 x + C_1^> & x < 0 \end{cases}$$

$$\Rightarrow \quad -\frac{d\phi}{dx} = \vec{E} = \begin{cases} (2\pi\sigma - \bar{C}_0)\,\hat{x} & x > 0 \\ (-2\pi\sigma - \bar{C}_0)\,\hat{x} & x < 0 \end{cases}$$

Const $C_1^>$ does not effect $\vec{E}$ – additive const to ϕ

$\bar{C}_0$ represents const uniform electric field $-\bar{C}_0\hat{x}$, that exists independently of the charged surface

If we assumed that all $\vec{E}$ fields are just those arising from the plane, then we can set $\bar{C}_0 = 0$. Equivalently, if the plane is the only source of $\vec{E}$, then we expect ϕ depends only on $|x|$ by symmetry. $\Rightarrow$ $C_0^< = -C_0^>$ and again $\bar{C}_0 = 0$. In this case

$$\phi(x) = \begin{cases} -2\pi\sigma x & x > 0 \\ 2\pi\sigma x & x < 0 \end{cases}$$

(we also set $C_1^> = 0$ here corresponding to $\phi(0) = 0$

$$\vec{E}(x) = \begin{cases} 2\pi\sigma\,\hat{x} & x > 0 \\ -2\pi\sigma\,\hat{x} & x < 0 \end{cases}$$

$\vec{E}$ is constant but oppositely directed on either side of the charged plane

Green's theorem, Uniqueness, Green function - part II

We want to show that the boundary value problem we described is well posed - ie there is a unique solution. We start by deriving Greens theorems

Consider $\int_V d^3r\, \vec{\nabla}\cdot\vec{A} = \oint_S da\, \hat{n}\cdot\vec{A}$ Gauss theorem

let $\vec{A} = \phi\vec{\nabla}\psi$ ϕ, ψ any two scalar functions

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{A} = \phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$$

$$\phi\vec{\nabla}\psi\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$

$$\Rightarrow \int_V d^3r\left(\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\, \phi\frac{\partial\psi}{\partial n}$$ Green's 1st identity

let $\phi \leftrightarrow \psi$

$$\int_V d^3r\left(\psi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\, \psi\frac{\partial\phi}{\partial n}$$

subtract

$$\int_V d^3r\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S da\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)$$ Green's 2nd identity

Apply Greens 2nd identity with $\psi = \frac{1}{|\vec{r}-\vec{r}'|}$, $\vec{r}'$ is integration variable, ϕ is the scalar potential with $\nabla^2\phi = -4\pi\rho$. use $\nabla^2\psi = \nabla'^2\psi = -4\pi\,\delta(\vec{r}-\vec{r}')$

$$\int_V d^3r'\left[\phi(r')\left[-4\pi\,\delta(r-r')\right] - \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\left(-4\pi\rho(\vec{r}')\right)\right]$$

$$= \oint_S da'\left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'}\right]$$

If $\vec{r}$ lies within the volume V, then

$$(*) \qquad \phi(\vec{r}) = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

Note: if $\vec{r}$ lies outside the volume V, then

$$0 = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

potential from a surface charge density $\sigma = \frac{1}{4\pi}\frac{\partial\phi}{\partial n'}$

potential from a surface dipole layer of dipole strength density $\frac{\phi}{4\pi}$

From $(*)$, if $S \to \infty$ and $E \sim \frac{\partial\phi}{\partial n} \to 0$ faster than $\frac{1}{r}$, then the surface integral vanishes and we recover Coulombs law $\phi(\vec{r}) = \int d^3r' \rho(\vec{r}')/|\vec{r}-\vec{r}'|$

$(*)$ gives the generalization of Coulombs law to a system with a finite boundary

For a <u>charge free</u> volume V, ie $\rho(r) = 0$ in V, the potential everywhere is determined by the potential and its normal derivative on the surface.

But one cannot in general freely specify <u>both</u> ϕ and $\frac{\partial\phi}{\partial n}$ on the boundary surface since the resulting ϕ from $(*)$ would not in general obey Laplace's equation $\nabla^2\phi = 0$.

Specifying both ϕ and $\frac{\partial \phi}{\partial n}$ on surface is known as "Cauchy" boundary conditions — for Laplace's equ, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

Uniqueness

If we have a system of charges in vol V, and either the potential ϕ, or its normal derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of V, then there is a unique solution to Poisson's equation inside V. Specifying ϕ is known as Dirichlet boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as Neumann boundary conditions.

proof: Suppose we had two solutions ϕ_1 and ϕ_2, both with $-\nabla^2 \phi = 4\pi\rho$ inside V, and obeying specified b.c. on surface of V.

Define $u = \phi_2 - \phi_1 \Rightarrow \nabla^2 u = 0$ inside V

and $u = 0$ on surface S - for Dirichlet b.c.
or $\frac{\partial u}{\partial n} = 0$ on surface S - for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = u$

$$\int_V d^3r \, (\underbrace{u \nabla^2 u}_{\substack{= \\ 0 \\ \text{as } \nabla^2 u = 0}} + \vec{\nabla} u \cdot \vec{\nabla} u) = \oint_S da \, \underbrace{u \frac{\partial u}{\partial n}}_{\substack{= \\ 0 \\ \text{as } u \text{ or } \frac{\partial u}{\partial n} = 0}}$$

$$\Rightarrow \int_V d^3r \, |\vec{\nabla} U|^2 = 0 \qquad \Rightarrow \vec{\nabla} U = 0$$
$$\Rightarrow U = \text{const}$$

For Dirichlet b.c., $U = 0$ on surface S, so const $= 0$ and $\phi_1 = \phi_2$. Solution is unique

For Neumann b.c., ϕ_1 and ϕ_2 differ only by an arbitrary constant. Since $\vec{E} = -\vec{\nabla}\phi$, the electric fields $\vec{E}_1 = -\vec{\nabla}\phi_1$ and $E_2 = -\vec{\nabla}\phi_2$ are the same.

~~[illegible]~~ If boundary ~~solution~~ surface S consists of several disjoint pieces, then solution is unique if specify ϕ on some pieces and $\frac{\partial \phi}{\partial n}$ on other pieces.

Solution of Poisson's equation with both ϕ and $\frac{\partial \phi}{\partial n}$ specified on the same surface S (Cauchy b.c.) does not in general exist, since specifying either ϕ or $\frac{\partial \phi}{\partial n}$ alone is enough to give a unique solution.

Green's function - part II

Greens 2nd identity

$$\int_V d^3r' \left(\phi \nabla'^2 \psi - \psi \nabla'^2 \phi \right) = \oint_S da' \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right)$$

Apply above with $\begin{cases} \phi(\vec{r}') \text{ electrostatic potential with } \nabla'^2\phi = -4\pi\rho(\vec{r}') \\ \psi(\vec{r}') = G(\vec{r},\vec{r}') \text{ the Green function satisfying} \end{cases}$

$$\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$$

we saw one solution of above is $G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$

but a more general solution is

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}')$$

where $\nabla'^2 F(\vec{r},\vec{r}') = 0$, for $\vec{r}'$ in volume V

we will choose $F(\vec{r},\vec{r}')$ to simplify solution of ϕ

$$\Rightarrow \int_V d^3r' \left(\phi(\vec{r}') \nabla'^2 G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}') \nabla'^2 \phi(\vec{r}') \right)$$

$$= \int_V d^3r' \left(\phi(r') \left[-4\pi\delta(\vec{r}-\vec{r}') \right] - G(\vec{r},\vec{r}') \left[-4\pi\rho(\vec{r}') \right] \right)$$

$$= -4\pi\phi(\vec{r}) + 4\pi \int_V d^3r' \, G(\vec{r},\vec{r}')\rho(\vec{r}')$$

$$= \oint_S da' \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

$$\phi(\vec{r}) = \int_V d^3r'\, G(\vec{r},\vec{r}')\, \rho(\vec{r}') + \oint_S \frac{da'}{4\pi} \left(G(\vec{r},\vec{r}') \frac{\partial \phi(\vec{r}')}{\partial n'} - \phi(r') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} \right)$$

Consider Dirichlet boundary problem. If we can choose $F(\vec{r},\vec{r}')$ such that $G_D(\vec{r},\vec{r}') = 0$ for $\vec{r}'$ on the boundary surface S, then above simplifies to

$$\boxed{\phi(\vec{r}) = \int_V d^3r'\, G_D(\vec{r},\vec{r}')\, \rho(\vec{r}') - \oint_S \frac{da'}{4\pi} \phi(r') \frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'}}$$

Since $\rho(r)$ is specified in V, and $\phi(r)$ is specified on S, above then gives desired solution for $\phi(r)$ inside volume V.

Finding G_D is therefore equivalent to finding an $F(r,r')$ such that $\nabla'^2 F(\vec{r},\vec{r}') = 0$ for $\vec{r}'$ in V (solves Laplace eqn) and

$$F(\vec{r},\vec{r}') = \frac{-1}{|\vec{r}-\vec{r}'|} \quad \text{for } \vec{r}' \text{ on boundary surface } S'$$

Always exists unique solution for F

Next consider <u>Neumann boundary problem</u>.

One might think to find $F(\vec{r},\vec{r}')$ such that $\frac{\partial G(\vec{r},\vec{r}')}{\partial n'} = 0$ on boundary surface. But this is not possible.

Consider $\int_V \nabla'^2 G(\vec{r},\vec{r}')\, d^3r' = \int \vec{\nabla}'\cdot\vec{\nabla}' G(\vec{r},\vec{r}')\, d^3r'$

$$= \oint_S \vec{\nabla}' G(\vec{r},\vec{r}')\cdot\hat{n}\, da'$$

$$= \oint_S \frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\, da' = -4\pi \qquad \text{since } \nabla'^2 G = -4\pi\delta(r-r')$$

So we can't have $\frac{\partial G}{\partial n'} = 0$ for $\vec{r}'$ on S

Simplest choice is then $\frac{\partial G_N(r,r')}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{r}'$ on S ← area of surface

Then

$$\phi(\vec{r}) = \int_V d^3r'\, G_N(\vec{r},\vec{r}')\,\rho(\vec{r}') + \oint \frac{da'}{4\pi} G(\vec{r},\vec{r}')\frac{\partial\phi(\vec{r}')}{\partial n'} + \oint \frac{da'}{4\pi}\phi(r')\left(\frac{-4\pi}{S}\right)$$

$$\left[\phi(\vec{r}') = \int_V d^3r'\, G_N(\vec{r},\vec{r}')\,\rho(\vec{r}') + \oint \frac{da'}{4\pi} G_N(\vec{r},\vec{r}')\frac{\partial\phi(\vec{r}')}{\partial n'}\right] + \langle\phi\rangle_S$$

constant = average value of ϕ on surface S.

Since $\rho(r)$ is specified in V and $\frac{\partial\phi}{\partial n}$ is specified on S

above gives solution $\phi(r)$ in V within additive constant $\langle\phi\rangle_S$

Since $\vec{E} = -\vec{\nabla}\phi$, the const $\langle\phi\rangle_S$ is of no consequence

Finding $G_N(\vec{r},\vec{r}')$ is therefore equivalent to finding an $F(\vec{r},\vec{r}')$ such that

$$\nabla'^2 F(\vec{r},\vec{r}') = 0 \quad \text{for } \vec{r}' \text{ in } V$$

$$\text{and} \quad \frac{\partial F(\vec{r},\vec{r}')}{\partial n'} = \frac{-4\pi}{S'} \quad \text{for } \vec{r}' \text{ on surface } S'$$

always exists a unique solution (within additive constant)

While G_D and G_N always exist in principle, they depend in detail on the shape of the surface S' and are difficult to find except for simple geometries

In preceding we defined G by $\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$

But our earlier interpretation of $G(\vec{r},\vec{r}')$ was that it was potential at $\vec{r}$ due to point source at $\vec{r}'$, ie $\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$. Note, for general surface S', $G(\vec{r},\vec{r}')$ is not in general a function of $|\vec{r}-\vec{r}'|$ but depends on $\vec{r}$ and $\vec{r}'$ separately. But the equivalence of the two definitions of G above is obtained by noting that one can prove the symmetry property

$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$$

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c.
(see Jackson, end section 1.10)