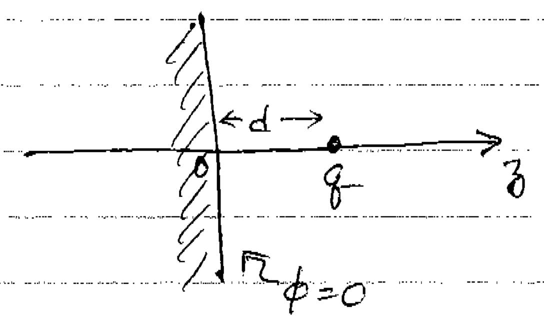
# Image Charge method

For simple geometries, can try to obtain $G_D$ or $G_N$ by placing a set of "image charges" outside the volume of interest $V$, i.e. on the "other side" of the system boundary surface $S$. Because these image charges are outside $V$, their contrib to the potential inside $V$ obeys $\nabla^2\phi^{\text{image}} = 0$, as necessary. Choose location of image charges so that total $\phi$ has desired boundary condition.

1) charge in front of infinite grounded plane




want $\nabla^2\phi = -4\pi q\,\delta(x)\delta(y)\delta(z-d)$

$\phi = 0$ for $z = 0$

If we find a solution to above it is the unique solution

Solution - put fictitious image charge $-q$ at $z = -d$
$\phi$ is Coulomb potential from the real charge + the image

$$\phi(\vec{r}) = \underbrace{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}}_{\text{real charge}} + \underbrace{\frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}}_{\text{image charge}}$$

above satisfies $\phi(x,y,0) = 0$ as required

Also, $\nabla^2\phi = -4\pi q\,\delta(\vec{r} - d\hat{z}) + 4\pi q\,\delta(\vec{r} + d\hat{z})$

$= -4\pi q\,\delta(\vec{r} - d\hat{z})$ for region $z > 0$

Can now find $\vec{E}$ for $z > 0$

$$\vec{E} = -\vec{\nabla}\phi$$

In particular $E_z = -\frac{\partial \phi}{\partial z} = +q\left[\left(\frac{1}{2}\right)\frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}}\right.$

$$\left. -\left(\frac{1}{2}\right)\frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

$$E_z = q\left[\frac{(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

We can use above to compute the surface charge density $\sigma(x,y)$ induced on the surface of the conducting plane. At conductor surface

$$-\frac{\partial \phi}{\partial n} = 4\pi\sigma$$

$$\Rightarrow \quad \sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial z} = \frac{1}{4\pi}E_z(x,y,z=0)$$

$$\sigma(x,y) = \frac{q}{4\pi}\left[\frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}}\right]$$

$$= \frac{-q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}} = \frac{-qd}{2\pi\,(r_\perp^2+d^2)^{3/2}}$$

$$r_\perp = \sqrt{x^2+y^2}$$

Total induced charge is

$$q^{induced} = \int_{-\infty}^{\infty} dx\,dy\ \sigma(x,y)$$

$$= 2\pi \int_0^{\infty} dr_\perp\, r_\perp\ \sigma(r_\perp)$$

$$= 2\pi \int_0^{\infty} dr_\perp \frac{r_\perp\ (-qd)}{2\pi\left(r_\perp^2+d^2\right)^{3/2}}$$

$$= -qd\left[\frac{-1}{(r_\perp^2+d^2)^{1/2}}\right]_0^{\infty}$$

$$= -qd\left[0 - \frac{-1}{d}\right]$$

$$q^{induced} = -q \qquad \text{induced charge} = \text{image charge}$$

Force on charge $q$ in front of conducting plane is due to the induced $\sigma$. The $E$ field of this $\sigma$ is, for $z>0$, the same as the $E$ field of the image charge.

$$\Rightarrow \vec{F} = \frac{-q^2}{(2d)^2}\hat{z} = \frac{-q^2}{4d^2}\hat{z} \qquad \underline{\text{attractive}}$$

Work done to move $q$ into position from infinity is

$$W = -\int_{\infty}^{d} d\vec{l}\cdot\vec{F} = -\int_{\infty}^{d} dz\, F_z$$

electrostatic force $\vec{F}$

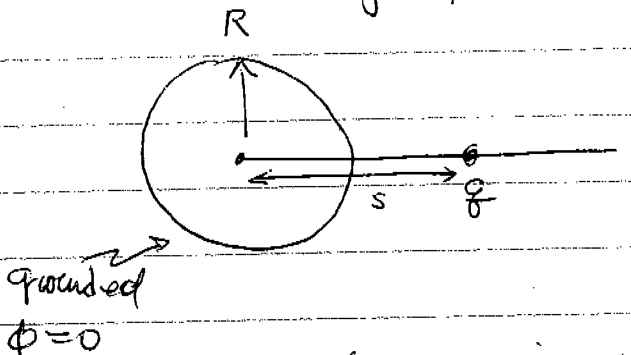
$$W = \int_d^\infty dz \left( \frac{-q^2}{4z^2} \right) = \frac{-q^2}{4d}$$

$W < 0 \Rightarrow$ energy released

Note: $W$ above is not the electrostatic energy that would be present if the image charge were real i.e. it is not $q\phi^{\text{image}}(\vec{r} = d\hat{z}) = \frac{-q^2}{2d}$

One way to see why is to note that as $q$ is moved quasistatically in towards the conducting plane, the image charge also must be moving to stay equidistant on the opposite side,

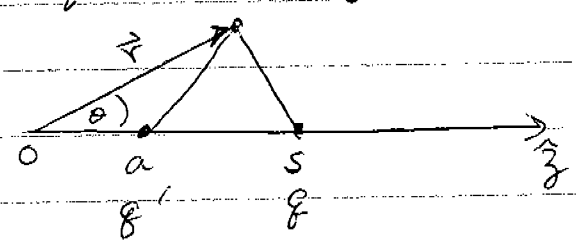
2) point charge in front of a grounded ($\phi = 0$) conducting sphere.




charge $q$ placed a distance $s$ from center of grounded conducting sphere of radius $R$

place image charge $q'$ inside sphere so that the combined $\Phi$ from $q$ and $q'$ vanishes on surface of sphere.

By symmetry, $q'$ should lie on the same radial line as $q$ does. Call the distance of $q'$ from the origin "$a$"




potential at position $\vec{r}$ is

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - s\hat{z}|} + \frac{q'}{|\vec{r} - a\hat{z}|}$$

$$= \frac{q}{(r^2 + s^2 - 2sr\cos\theta)^{1/2}} + \frac{q'}{(r^2 + a^2 - 2ra\cos\theta)^{1/2}}$$

Can we choose $q'$ and $a$ so that $\phi(R, \theta) = 0$ for all $\theta$?

$$\phi(R,\theta) = \frac{q}{(R^2+s^2-2sR\cos\theta)^{1/2}} + \frac{q'}{(R^2+a^2-2aR\cos\theta)^{1/2}}$$

make denominators look alike

$$R^2+a^2-2aR\cos\theta = \frac{a}{s}\left(\frac{s}{a}R^2 + sa - 2sR\cos\theta\right)$$

if choose $sa = R^2$, ie $\boxed{a = R^2/s}$, then $\frac{sR^2}{a} = s^2$
and then the denominator of the 2nd term is

$$\left[\frac{R^2}{s^2}(s^2+R^2-2sR\cos\theta)\right]^{1/2} = \frac{R}{s}\left[s^2+R^2-2sR\cos\theta\right]^{1/2}$$

$$\Rightarrow \quad \phi(R,\theta) = \frac{q}{(R^2+s^2-2sR\cos\theta)^{1/2}} + \frac{q'(s/R)}{(R^2+s^2-2sR\cos\theta)^{1/2}}$$

So choose $q'(s/R) = -q \rightarrow \boxed{q' = -q\,R/s}$
to get $\phi(R,\theta) = 0$

Solution is

$$\phi(r,\theta) = \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{q\,R/s}{\left(r^2+\frac{R^4}{s^2}-2r\frac{R^2}{s}\cos\theta\right)^{1/2}}$$

$$= \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{q}{\left(\frac{s^2r^2}{R^2}+R^2-2rs\cos\theta\right)^{1/2}}$$

Can get induced surface charge on sphere by

$4\pi\sigma = \vec{E}\cdot\hat{r} = -\left.\frac{\partial\phi}{\partial r}\right|_{r=R}$ see Jackson Eq (2.5) for result

$$\sigma(\theta) = -\frac{q}{4\pi}\frac{1}{Rs}\frac{1-(R/s)^2}{\left(1+(R/s)^2-2(R/s)\cos\theta\right)^{3/2}}$$

$\sigma(\theta)$ is greatest at $\theta=0$, as one should expect

Can integrate $\sigma(\theta)$ to get total induced charge. One finds

$$2\pi\int_0^\pi d\theta \sin\theta\, R^2\,\sigma(\theta) = q' = -q\,R/s$$

In general, total induced charge = sum of all image charges

<u>Force of attraction</u> of charge to sphere

Force on $q$ is due to electric field from induced charge $\sigma$ which is the same as the electric field from the image charge $q'$.

$$\vec{F} = \frac{qq'\,\hat{z}}{(s-a)^2} = \frac{-q^2(R/s)\,\hat{z}}{(s-R^2/s)^2} = \frac{-q^2Rs}{(s^2-R^2)^2}\,\hat{z}$$

<u>Close to the surface</u> of the sphere, $s\approx R$, so write $s=R+d$ where $d\ll R$. Then

$$\vec{F} = \frac{-q^2Rs}{(s-R)^2(s+R)^2} = \frac{-q^2R(R+d)}{d^2(2R+d)^2} \simeq \frac{-q^2}{4d^2}$$

get same result as for infinite flat grounded plane. When $q$ is so close to surface that $d\ll R$, the charge does not "see" the curvature of the surface far from the surface, $s \gg R$

$$\vec{F} = \frac{q q' \hat{z}}{(s-a)^2} = \frac{-q^2 R s}{(s^2 - R^2)^2}\,\hat{z} \simeq -\frac{q^2 R}{s^3}\,\hat{z}$$

$F \sim \frac{1}{s^3}$ very different from flat plane
also different from point charge

Note: In preceding two problems, what we found was a $\phi$ such that $\nabla^2 \left(\frac{\phi}{q}\right) = -4\pi\delta(\vec{r}-\vec{r}_0)$, for a charge at $\vec{r}_0$, and $\phi = 0$ on the boundary. Such a $\phi$ is nothing more than $G_D$ the corresponding Green function for Dirichlet boundary conditions.

Suppose now that instead of a grounded sphere we have a sphere with fixed net charge $Q$.

We want to add new image charge to represent this case. If we put $q' = -qR/s$ at $a = R/s$ as before, the boundary condition of $\phi = \text{const}$ on surface $r = R$ is met, but the net charge on the sphere is $q'$ (the induced charge) not the desired $Q$. We therefore need to add new image charge(s) of total charge $Q - q'$ (so total image charge is $Q$) in such a way that we keep $\phi$ constant on the surface of the sphere. The way to do this is to put $Q - q'$ at the origin!

Solution is

$$\Phi(r,\theta) = \frac{Q + qR/s}{r} + \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{q}{\left(\frac{s^2r^2}{R^2}+R^2-2rs\cos\theta\right)^{1/2}}$$

The force on the charge $q$ is due to the $\vec{E}$ field of the images

$$\vec{F} = F\hat{z} = \frac{q(Q+qR/s)\,\hat{z}}{s^2} + \frac{q q' \,\hat{z}}{(s-a)^2}$$

$$F = \frac{qQ}{s^2} + \frac{q^2R/s}{s^2} - \frac{q^2R/s}{(s-R^2/s)^2}$$

$$= \frac{qQ}{s^2} + q^2R\left[\frac{1}{s^3} - \frac{1}{s^3(1-R^2/s^2)^2}\right]$$

$$= \frac{qQ}{s^2} + \frac{q^2R}{s^3}\left[1 - \frac{1}{(1-R^2/s^2)^2}\right]$$

$$F = \frac{qQ}{s^2} - q^2\frac{R^3}{s}\frac{2-R^2/s^2}{(s^2-R^2)^2}$$

For large $s \gg R$ far from surface

$$F \simeq \frac{qQ}{s^2} - \frac{2q^2R^3}{s^5}$$

leading term is just Coulomb force between $q$ and $Q$ at origin

for $Q>0$, $F$ is always repulsive for large enough $s$

For $s = R+d$, $d \ll R$ close to surface

$$F = \frac{qQ}{(R+d)^2} - \frac{q^2 R^3}{(R+d)} \frac{2 - \frac{R^2}{(R+d)^2}}{(R^2+d^2+2Rd-R^2)^2}$$

$$\simeq \frac{qQ}{R^2} - q^2\frac{R^3}{R} \frac{(2-1)}{4R^2 d^2}$$

$$F \simeq \frac{qQ}{R^2} - \frac{q^2}{4d^2} \simeq \frac{-q^2}{4d^2} \quad \text{for } d \text{ small enough}$$

$F$ is always attractive for small enough $d$, and is equal to the force in front of a grounded plane, no matter what is the value of $Q$! This is because the image charge $q'$ lies so much closer to $q$ than does the $Q-q'$ at the origin, that it dominates the force.

The cross over from attractive to repulsive occurs at a distance $s$ that depends on $Q$. This distance is given by

$$\frac{Q}{q} = \frac{R^3 s\,(2 - R^2/s^2)}{(s^2-R^2)^2} = \left(\frac{R}{s}\right)^3 \frac{2-(R/s)^2}{[1-(R/s)^2]^2}$$

let $x = R/s \in (0,1)$

$$\frac{Q}{q} = x^3 \frac{(2-x^2)}{(1-x^2)^2}$$

gives $5^{th}$ order polynomial in $x$
no analytic solution
can solve graphically

For $\frac{Q}{q} = 1$, cross over is at $\frac{R}{s} = 0.62$

$s = 1.6R$

$\frac{Q}{q} = 0.1$ cross over is at $\frac{R}{s} = 0.36$

$s = 2.8R$