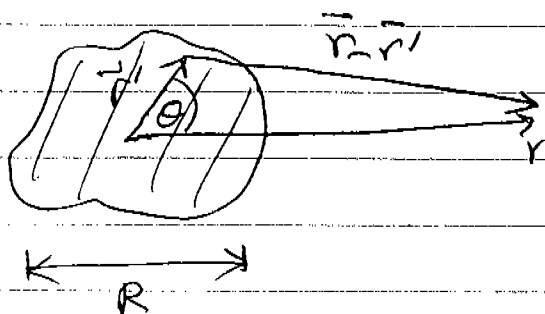


Multipole Expansion

region with $\rho \neq 0$



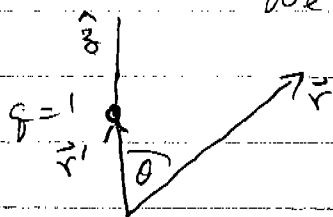
We want to find the potential ϕ for an arbitrary localized distribution of charge ρ , at distances far away $r \gg R$.

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

General Coulomb formula

We want an expansion of $\frac{1}{|\vec{r} - \vec{r}'|}$ in powers of $\left(\frac{r'}{r}\right)$ for $r \gg r'$

$\frac{1}{|\vec{r} - \vec{r}'|}$ view this as the potential at $\vec{r}$ due to a unit point charge located at position $\vec{r}'$. We take $\vec{r}'$ on the $\hat{z}$ axis.



The problem has azimuthal symmetry $\Rightarrow \phi$ depends only on r and θ , so we can express it as an expansion in Legendre polynomials.

For $r \gg r'$,

$$\phi(r, \theta) = \sum_{\ell=0}^{\infty} \frac{B_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta)$$

all $A_{\ell} = 0$
as need $\phi \rightarrow 0$
as $r \rightarrow \infty$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_{\ell}}{r^{\ell}} P_{\ell}(\cos \theta)$$

We know $\phi(r, \theta=0) = \frac{1}{r-r'}$ (for $r > r'$)

← scalars here since when $\theta=0$, $\vec{r}$ and $\vec{r}'$ are both on $\hat{z}$ axis

$$\Rightarrow \phi(r, 0) = \frac{1}{r} \sum_l \frac{B_l}{r^l} P_l(1)$$

$$= \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^l} \quad \text{as } P_l(1) = 1$$

$$= \frac{1}{r} \frac{1}{(1 - r'/r)} \leftarrow \text{exact result from Coulomb}$$

Now Taylor expansion $\frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \dots$

$$\Rightarrow \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^l} = \frac{1}{r} \left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^3 + \dots \right)$$

$$\Rightarrow B_l = (r')^l \text{ is solution}$$

So for $r > r'$

$$\boxed{\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos \theta)}$$

So for the charge distribution ρ ,

$$\begin{aligned} \phi(\vec{r}) &= \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} = \int d^3r' \frac{\rho(\vec{r}')}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos \theta) \\ &= \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int d^3r' \rho(\vec{r}') (r')^l P_l(\cos \theta) \end{aligned}$$

where θ is the angle between the fixed observation point $\vec{r}$ and the integration variable $\vec{r}'$.

This is the multipole expansion, which expresses the potential far from a localized source as a power series in (r'/r) . It is exact provided one adds all the infinite l terms. In practice, one generally approximates by summing only up to some finite l .

Note: in doing the integrals

$$\int d^3r' \rho(\vec{r}') (r')^l P_l(\cos\theta)$$

θ is defined as the angle of $\vec{r}'$ with respect to observation point $\vec{r}$. We therefore in principle have to repeat the integration every time we change $\vec{r}$.

We will find a way around this by

- (i) just looking explicitly at the few lowest order terms
- (ii) a general method involving spherical harmonics $Y_{lm}(\theta, \phi)$

monopole: $l=0$ term

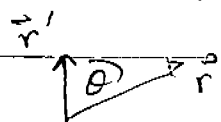
$$\phi^{(0)}(\vec{r}) = \frac{1}{r} \int d^3r' f(r') \quad P_0(\cos\theta) = 1$$

$$= \frac{Q}{r} \quad \text{where } Q \equiv \int d^3r' f(r') \text{ is}$$

total charge

dipole: $l=1$ term

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2} \int d^3r' f(\vec{r}') r' P_1(\cos\theta)$$



$$= \frac{1}{r^2} \int d^3r' f(\vec{r}') r' \cos\theta$$

$$\text{Now } \hat{r} \cdot \vec{r}' = r r' \cos\theta \Rightarrow \hat{r} \cdot \vec{r}' = r' \cos\theta$$

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2} \hat{r} \cdot \int d^3r' f(\vec{r}') \vec{r}'$$

$$= \frac{\vec{P} \cdot \hat{r}}{r^2} \quad \text{where } \vec{P} \equiv \int d^3r' f(\vec{r}') \vec{r}'$$

is the dipole moment

For a set of point charges q_i at $\vec{r}_i$,

$$\vec{P} = \sum q_i \vec{r}_i$$

quadrupole: $l=2$ term

$$\begin{aligned}\phi^{(2)}(\vec{r}) &= \frac{1}{r^3} \int d^3r' \rho(\vec{r}') r'^2 P_2(\cos\theta) \\ &= \frac{1}{r^3} \int d^3r' \rho(\vec{r}') r'^2 \frac{1}{2} (3\cos^2\theta - 1)\end{aligned}$$

we $\cos\theta = \hat{r}' \cdot \hat{r}$

$$\begin{aligned}\phi^{(2)}(\vec{r}) &= \frac{1}{r^3} \int d^3r' \rho(\vec{r}') \frac{1}{2} (3(\hat{r}' \cdot \hat{r})^2 - (r')^2) \\ &= \frac{1}{r^3} \hat{r} \cdot \left[\int d^3r' \rho(\vec{r}') \frac{1}{2} (3\vec{r}'\vec{r}' - (r')^2 \vec{I}) \right] \cdot \hat{r}\end{aligned}$$

where $\vec{I}$ is the identity tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u} \cdot \vec{I} \cdot \vec{v} = \vec{u} \cdot \vec{v}$

and $\vec{r}'\vec{r}'$ is the tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u} \cdot [\vec{r}'\vec{r}'] \cdot \vec{v} = (\vec{u} \cdot \vec{r}')(\vec{r}' \cdot \vec{v})$

Define quadrupole tensor $\vec{Q} \equiv \int d^3r' \rho(\vec{r}') (3\vec{r}'\vec{r}' - (r')^2 \vec{I})$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3} \frac{1}{2} \hat{r} \cdot \vec{Q} \cdot \hat{r}$$

So to lowest three terms

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot \vec{Q} \cdot \hat{r}}{2r^3} + \dots$$

defined in terms of the moments q , $\vec{p}$, $\vec{Q}$ of the charge distribution.

Note, the moments q , $\vec{p}$, $\vec{Q}$ do not depend on the observation point $\vec{r}$ - we can calculate them once and then use them to get $\phi(\vec{r})$ at all $\vec{r}$.

monopole: $q = \int d^3r \rho(\vec{r})$ scalar integral

dipole: $\vec{p} = \int d^3r \rho(\vec{r}) \vec{r}$ vector integral
 $\hat{e}_1 \equiv \hat{x}, \hat{e}_2 \equiv \hat{y}, \hat{e}_3 \equiv \hat{z}$

if we pick a coordinate system, we have to do 3 integrations to get the three components of $\vec{p}$
 $\hat{e}_i \cdot \vec{p} = p_i = \int d^3r \rho(\vec{r}) r_i$

quadrupole: $\vec{Q} = \int d^3r \rho(\vec{r}) (3\vec{r} \vec{r} - r^2 \vec{I})$ tensor integral

if we pick a coord system x, y, z then

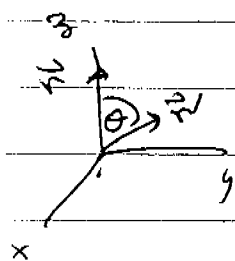
$\vec{Q}$ is a matrix with components $\hat{e}_1 \equiv \hat{x}, \hat{e}_2 \equiv \hat{y}, \hat{e}_3 \equiv \hat{z}$

$$\hat{e}_i \cdot \vec{Q} \cdot \hat{e}_j = Q_{ij} = \int d^3r \rho(\vec{r}) [3r_i r_j - r^2 \delta_{ij}]$$

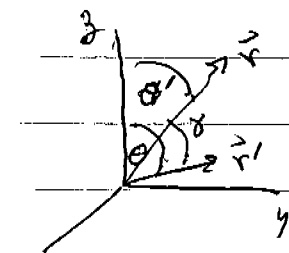
There are 9 elements of the 3×3 matrix Q_{ij} , but $Q_{ij} = Q_{ji}$ is symmetric so there are only 6 independent elements to compute.

General method

$$\phi(\vec{r}) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int d^3r' \rho(\vec{r}') (r')^l P_l(\cos\theta)$$



in above, θ is angle between $\vec{r}$ and $\hat{z}$
 if we think of $P_l \cos\theta$ as the spherical coord θ ,
 then in effect, above is choosing $\vec{r}$ to be on
 $\hat{z}$ axis. We would like a representation in
 which $\vec{r}$ is positioned arbitrarily with respect
 to the axes used in describing ρ



Use the addition theorem for spherical harmonics
 - see Jackson 3.6 for discussion & proof

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

where (θ, ϕ) are the angles of $\hat{r}$, (θ', ϕ') are
 the angles of $\hat{r}'$, and γ is the angle
 between $\hat{r}$ and $\hat{r}'$, i.e. $\cos\gamma = \hat{r} \cdot \hat{r}'$

$$\cos\theta = \hat{z} \cdot \hat{r}$$

$$\cos\theta' = \hat{z} \cdot \hat{r}'$$

$\Rightarrow$

$$\phi(\vec{r}) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \frac{4\pi}{2l+1} \sum_{m=-l}^l \int d^3r' \rho(\vec{r}') (r')^l Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

Define the moment

$$Q_{lm} \equiv \int d^3r' \rho(\vec{r}') (r')^l Y_{lm}^*(\theta', \phi')$$

independent of observation point

Then

$$\phi(\vec{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{g_{lm} Y_{lm}(\theta, \phi)}{(2l+1) r^{l+1}}$$

see Jackson equ (4.4), (4.5), (4.6) to relate g_{lm} to $q, \vec{p}, \vec{Q}$.

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot \vec{Q} \cdot \hat{r}}{2r^3}$$

electric field $\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial\phi}{\partial\theta} \hat{\theta} + \frac{1}{r\sin\theta} \frac{\partial\phi}{\partial\varphi} \hat{\varphi}$

For the monopole term $\vec{E} = \frac{q}{r^2} \hat{r}$

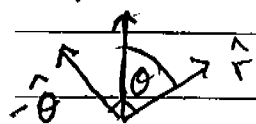
For the dipole term, choose $\vec{p}$ along $\hat{z}$ axis so

$$\phi(\vec{r}) = \frac{p \cos\theta}{r^2}$$

$$\vec{E} = \frac{2p \cos\theta}{r^3} \hat{r} + \frac{p \sin\theta}{r^3} \hat{\theta}$$

$$\vec{E} = \frac{p}{r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})$$

note $p \cos\theta \hat{r} = (\vec{p} \cdot \hat{r}) \hat{r}$
 $p \sin\theta \hat{\theta} = -(\vec{p} \cdot \hat{\theta}) \hat{\theta}$



Now $\vec{p} = (\vec{p} \cdot \hat{r}) \hat{r} + (\vec{p} \cdot \hat{\theta}) \hat{\theta}$

$$\Rightarrow -(\vec{p} \cdot \hat{\theta}) \hat{\theta} = (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}$$

So

$$\vec{E} = \frac{1}{r^3} [2(\vec{p} \cdot \hat{r}) \hat{r} + (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}]$$

$$= \frac{1}{r^3} [3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}]$$

expresses $\vec{E}$ in coord free form