

Boundary value problems in magnetostatics

Scalar Magnetic Potential

Because of the vector character of the equation

$$-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}$$

and the fact that $\nabla^2 \vec{A}$ only has a convenient representation in Cartesian coordinates, many of the methods we used to solve the scalar $-\nabla^2 \phi = 4\pi \rho$ don't work so well for magnetostatics.

However, in situations where the current $\vec{j}$ is confined to certain surfaces, we can make things much closer to the electrostatic case by using the trick of the scalar magnetic potential ϕ_M .

In regions where $\vec{j} = 0$, i.e. not on the certain surfaces, we have $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$. Since $\vec{\nabla} \times \vec{B} = 0$ in these regions, we can define a scalar potential ϕ_M such that

$$\vec{B} = -\vec{\nabla} \phi_M$$

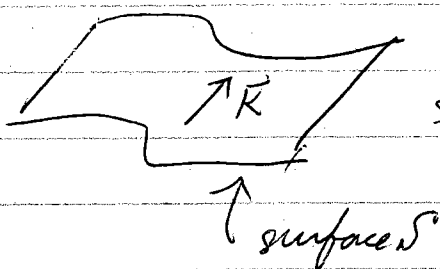
and then

$$\vec{\nabla} \cdot \vec{B} = -\nabla^2 \phi_M = 0$$

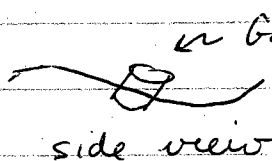
We can solve for ϕ_M as in electrostatics, and match solutions by applying appropriate boundary conditions on the current carrying surfaces.

Boundary Conditions at sheet current

in magnetostatics $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$



surface current $\vec{K}(\vec{r})$ at pt $\vec{r}$
on surface S

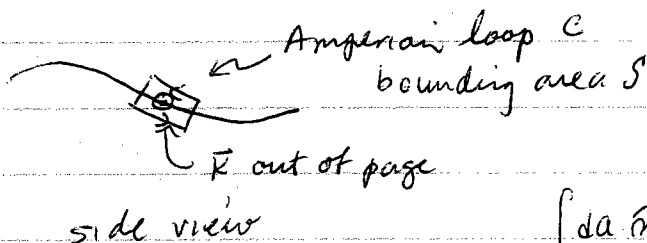


$$\int_V d^3r \vec{\nabla} \cdot \vec{B} = 0$$

top + bottom area of pill box is da
width of pill box $\rightarrow 0$

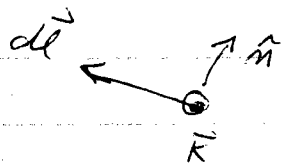
$$\Rightarrow \int_V d^3r \vec{\nabla} \cdot \vec{B} = \oint_S da \hat{n} \cdot \vec{B} = da (\vec{B}_{above} - \vec{B}_{below}) \cdot \hat{n} = 0$$

normal component of $\vec{B}$ is continuous $(\vec{B}_{above} - \vec{B}_{below}) \cdot \hat{n} = 0$



$$\oint_S da \hat{n} \cdot (\vec{\nabla} \times \vec{B}) = \oint_C \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c} I_{enclosed}$$

let width of loop $\rightarrow 0$, top + bottom sides $d\vec{\ell}$



$$(\vec{B}_{above} - \vec{B}_{below}) \cdot d\vec{\ell} = \frac{4\pi}{c} (\vec{\ell} \times \vec{K}) \cdot \hat{n} \\ = \frac{4\pi}{c} (\vec{K} \times \hat{n}) \cdot d\vec{\ell}$$

$\hat{n}$ is outward normal

tangential component of $\vec{B}$ has discontinuous jump $\frac{4\pi}{c} \vec{K} \times \hat{n}$

Combine both results into

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

magnetic analog of $\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = 4\pi\sigma \hat{n}$

In terms of magnetic vector potential ϕ_M

$$-\vec{\nabla}\phi_{M\text{above}} + \vec{\nabla}\phi_{M\text{below}} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

Note: ϕ_M is a calculational tool only
it does not have any direct physical
significance as does the electrostatic ϕ .
Electrostatic ϕ is related to work done
moving a charge $W_{12} = q[\phi(r_2) - \phi(r_1)]$
nothing similar for ϕ_M .

(in fact magnetostatic magnetic forces do no work!)

$$\begin{aligned}\vec{F} &= q \vec{v} \times \vec{B} \\ \Rightarrow \vec{F} \cdot \vec{v} &= \frac{dW}{dt} = 0\end{aligned}$$

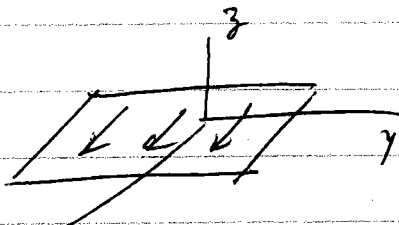
Note: we cannot apply argument ^{like} $\phi(r) - \phi(r') = \int_{r'}^r \vec{E} \cdot d\vec{l}$
 ϕ_M is not necessarily continuous at surface current
Cannot do similar to electrostatics and use
 $\phi_M(r_{\text{above}}) - \phi_M(r_{\text{below}}) = - \int_{r_{\text{below}}}^{r_{\text{above}}} \vec{B} \cdot d\vec{l}$

Since ϕ_M is not defined on the current sheet
itself, separating "above" from "below".

example

Flat infinite plane at $z=0$ with surface current

$$\vec{K} = K \hat{x}$$



x

$$z > 0, \nabla^2 \phi_M^> = 0 \Rightarrow \phi_M^> = a^> - b_x^> x - b_y^> y - b_z^> z$$

$$z < 0, \nabla^2 \phi_M^< = 0 \Rightarrow \phi_M^< = a^< - b_x^< x - b_y^< y - b_z^< z$$

$$z > 0, \vec{B}^> = -\vec{\nabla} \phi_M^> = b_x^> \hat{x} + b_y^> \hat{y} + b_z^> \hat{z}$$

$$z < 0, \vec{B}^< = -\vec{\nabla} \phi_M^< = b_x^< \hat{x} + b_y^< \hat{y} + b_z^< \hat{z}$$

$$\begin{aligned} \text{at } z=0 \quad \vec{B}^> - \vec{B}^< &= (b_x^> - b_x^<) \hat{x} + (b_y^> - b_y^<) \hat{y} + (b_z^> - b_z^<) \hat{z} \\ &= \frac{4\pi K}{c} \hat{x} \times \hat{z} = \frac{4\pi K}{c} (\hat{x} \times \hat{z}) = -\frac{4\pi K}{c} \hat{y} \end{aligned}$$

$$\Rightarrow b_x^> = b_x^< \equiv b_{x0}, \quad b_z^> = b_z^< \equiv b_{z0}, \quad b_y^> - b_y^< = -\frac{4\pi K}{c}$$

$$\begin{aligned} \text{define } \left. \begin{aligned} b_y^> &= b_{y0} + \delta b_y \\ b_y^< &= b_{y0} - \delta b_y \end{aligned} \right\} \delta b_y &= -\frac{2\pi K}{c} \end{aligned}$$

$$\Rightarrow \vec{B}^> = \vec{B}_0 - \frac{2\pi K}{c} \hat{y}$$

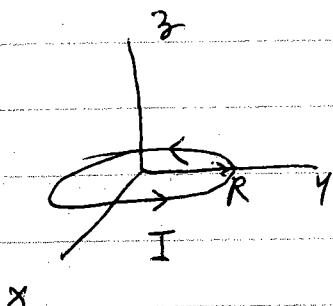
$$\vec{B}^< = \vec{B}_0 + \frac{2\pi K}{c} \hat{y}$$

$$\vec{B}_0 = b_{x0} \hat{x} + b_{y0} \hat{y} + b_{z0} \hat{z}$$

if $\vec{K}$ is the only source of magnetic field then $\vec{B}_0 = 0$

$$\vec{B} = \begin{cases} -\frac{2\pi K}{c} \hat{y} & z > 0 \\ \frac{2\pi K}{c} \hat{y} & z < 0 \end{cases}$$

example circular current loop in xy plane
radius R



for $r > R$, $\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla} \Phi_M$
where $\nabla^2 \Phi_M = 0$.

Try Legendre polynomial expansion for Φ_M

$$\Phi_M = \sum_{\ell=0}^{\infty} \frac{B_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta) \quad (A_{\ell} \text{ terms vanish as want } B \rightarrow 0 \text{ as } r \rightarrow \infty)$$

$$\vec{B} = -\vec{\nabla} \Phi_M = -\frac{\partial \Phi_M}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \Phi_M}{\partial \theta} \hat{\theta}$$

$$= \sum_{\ell} \left[\frac{(\ell+1)B_{\ell}}{r^{\ell+2}} P_{\ell}(\cos \theta) \hat{r} - \frac{B_{\ell}}{r^{\ell+2}} \frac{\partial P_{\ell}(\cos \theta)}{\partial \theta} \hat{\theta} \right]$$

write $\frac{\partial P_{\ell}}{\partial \theta} = \frac{\partial P_{\ell}}{\partial x} \frac{\partial x}{\partial \theta} = -\frac{\partial P_{\ell}}{\partial x} \sin \theta$ $x = \cos \theta$
 $\equiv -P'_{\ell} \sin \theta$

$$\vec{B} = \sum_{\ell} \left[\frac{(\ell+1)B_{\ell}}{r^{\ell+2}} P_{\ell}(\cos \theta) \hat{r} + \frac{B_{\ell}}{r^{\ell+2}} \sin \theta P'_{\ell}(\cos \theta) \hat{\theta} \right]$$

To determine the B_{ℓ} we compare with exact solution along $\hat{z}$ axis

$$\vec{B}(z\hat{z}) = \sum_{\ell} \frac{(\ell+1)B_{\ell}}{r^{\ell+2}} \hat{r} = \sum_{\ell} \frac{(\ell+1)B_{\ell}}{z^{\ell+2}} \hat{z}$$

since $P_{\ell}(1)=1$, $\sin(0)=0$ and $P'_{\ell}(1)$ finite, $\hat{r} = \hat{z}$ when $\theta=0$

exact solution on $\hat{z}$ axis:

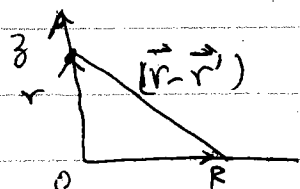
$$\vec{A} = \int \frac{d^3 r'}{c} \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A} = \int \frac{d^3 r'}{c} \vec{\nabla} \times \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\vec{B} = - \int \frac{d^3 r'}{c} \vec{j}(\vec{r}') \times \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$$

$$\vec{B} = \int \frac{d^3 r'}{c} \vec{j}(\vec{r}') \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Biot-Savart Law for magnetostatics

For our loop



$$\vec{B}(z) = \int_0^{2\pi} d\phi \frac{R}{c} I \hat{\phi} \times \frac{[-R \hat{r} + z \hat{z}]}{(z^2 + R^2)^{3/2}}$$

$$\hat{r} \times \hat{\phi} = \hat{z}$$

$$= \int_0^{2\pi} d\phi \frac{R(I R)}{c} \frac{\hat{z}}{(z^2 + R^2)^{3/2}}$$

$$\boxed{\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c (z^2 + R^2)^{3/2}}}$$

to match Legendre polynomial expansion, do Taylor series expansion of above

$$\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c z^3} \frac{1}{(1 + (\frac{R}{z})^2)^{3/2}} = \frac{2\pi R^2 I \hat{z}}{c z^3} \left\{ 1 - \frac{3}{2} \left(\frac{R}{z} \right)^2 + \dots \right\}$$

$$= \frac{2\pi R^2 I \hat{z}}{c} \left\{ \frac{1}{z^3} - \frac{3}{2} \frac{R^2}{z^5} + \dots \right\}$$

$$= \left\{ \frac{B_0}{z^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \dots \right\} \hat{z}$$

$$\Rightarrow B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3}{4c} \pi R^2 I R^2$$

So to order $l=3$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 P_1(\cos\theta) \hat{r} + \sin\theta P_1'(\cos\theta) \hat{\theta}}{r^3} - \left[\frac{3 R^2 P_3(\cos\theta) \hat{r} + \frac{3}{4} R^2 \sin\theta P_3'(\cos\theta) \hat{\theta}}{r^5} \right] + \dots \right\}$$

$$P_1(x) = x \Rightarrow P_1'(x) = 1$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \frac{1}{2}(15x^2 - 3)$$

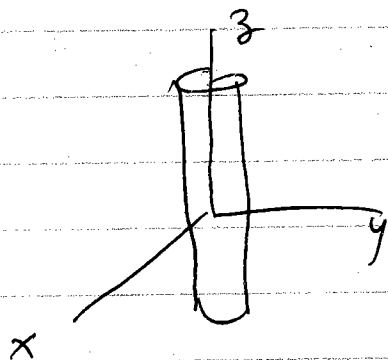
$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 \cos\theta \hat{r} + \sin\theta \hat{\theta}}{r^3} - \left[\frac{\frac{3}{2} R^2 (5 \cos^3\theta - 3 \cos\theta) \hat{r} + \frac{3}{8} R^2 \sin\theta (15 \cos^2\theta - 3) \hat{\theta}}{r^5} \right] + \dots \right\}$$

$\frac{\pi R^2 I}{c} = m$ is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic ^{octapole} ~~quadrupole~~ term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5.40)

example current carrying infinite cylinder radius R



- (i) $\vec{K} = K \hat{z}$ wire with surface current
 (ii) $\vec{K} = K \hat{\phi}$ solenoid

(c) $\vec{K} = K \hat{z}$ $2\pi R K = I$ total current

$r > R$ $\boxed{\Phi_M = -\frac{4\pi R K}{c} \varphi}$ magnetic scalar potential $\nabla^2 \Phi_M = 0$
 $r < R$ $\Phi_M = 0$

$r > R$ $\vec{B} = -\vec{\nabla} \Phi_M = -\frac{1}{r} \frac{\partial \Phi_M}{\partial \varphi} \hat{\phi} = \frac{4\pi R K}{c r} \hat{\phi} = \boxed{\frac{2I}{c r} \hat{\phi}}$ ← familiar result from Ampere
 $r < R$ $\vec{B} = 0$

$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \frac{2I}{c R} \hat{\phi} = \frac{4\pi K}{c} \frac{R}{R} \hat{\phi} = \frac{4\pi \vec{K}}{c} \times \hat{n}$
 where $\hat{n} = \hat{r}$
 as $\hat{z} \times \hat{r} = \hat{\phi}$

Note: $\Phi_M = -\frac{4\pi R K}{c} \varphi$ is not single valued!
 would not have found this using expansion of separation of coords in polar coords

(iii) $\vec{K} = K \hat{\phi}$

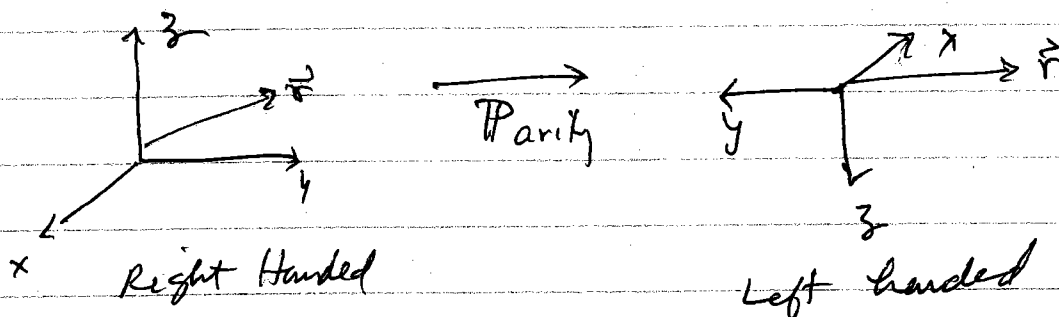
$r > R$ $\Phi_M = -B_1 z$
 $r < R$ $\Phi_M = -B_2 z$ $\left. \vphantom{\begin{matrix} r > R \\ r < R \end{matrix}} \right\} \nabla^2 \Phi_M = 0$
 $r > R$ $\vec{B} = -\vec{\nabla} \Phi_M = B_1 \hat{z}$
 $r < R$ $\vec{B} = -\vec{\nabla} \Phi_M = B_2 \hat{z}$

$$\begin{aligned}\vec{B}_{\text{above}} - \vec{B}_{\text{below}} &= (B_1 - B_2) \hat{z} = \frac{4\pi}{c} \vec{K} \times \hat{n} \\ &= \frac{4\pi}{c} K (\hat{\phi} \times \hat{r}) \\ &= -\frac{4\pi}{c} K \hat{z}\end{aligned}$$

If current in solenoid is only source of $\vec{B}$ Then expect $B_1 = 0$

$$\Rightarrow \boxed{\vec{B}_2 = \frac{4\pi}{c} K \hat{z}} \quad \text{familiar result}$$

Symmetry under parity transformation vector vs. pseudo vector



$$\vec{r} = (x, y, z) \rightarrow (-x, y, z)$$

$$P(\vec{r}) = -\vec{r} \quad \text{position } \vec{r} \text{ is odd under parity}$$

Any vector-like quantity that is odd under P is a vector.

examples of vectors

position $\vec{r}$

velocity $\vec{v} = \frac{d\vec{r}}{dt}$

acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

since $\vec{r}$ is vector and t is scalar
 $P(t) = t$

Force $\vec{F} = m\vec{a}$

momentum $\vec{p} = m\vec{v}$

since $\vec{a}$ is vector and m is scalar

since $\vec{v}$ is vector and m is scalar

electric field $\vec{E} = \vec{F}$

since $\vec{F}$ is vector and q is scalar
 $P(q) = q$

current $\vec{j} = \sum_i q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i(t))$

any vector-like quantity that is even under $\mathcal{P}$ is a pseudovector

angular momentum $\vec{L} = \vec{r} \times \vec{p}$ since $\vec{r} \rightarrow -\vec{r}$ and $\vec{p} \rightarrow \vec{p}$,
 $\vec{L} \rightarrow \vec{L}$ under $\mathcal{P}$

$\vec{L}$ is even under $\mathcal{P}$

magnetic field $\vec{F} = q \vec{v} \times \vec{B}$ since $\vec{F}$ and $\vec{v}$ are vectors and
 q is scalar, $\vec{B}$ must be pseudovector.

cross product of any two vectors is a pseudovector
 " " " vector and pseudovector is a vector

when solving for $\vec{E}$, it can only be made up of vectors that exist in the problem

When solving for $\vec{B}$, it can only be made up of pseudovectors that exist in the problem

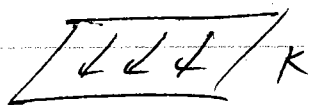
ex charged plane



only directions in problem is normal $\hat{n}$
 $\hat{n}$ is a vector

$$\vec{E} \propto \hat{n}$$

surface current



only directions are the vectors $\hat{n}$ and $\vec{K}$. But $\vec{B}$ can only be made of pseudovectors

$$\Rightarrow \vec{B} \propto (\vec{K} \times \hat{n})$$