Magnetic Materials

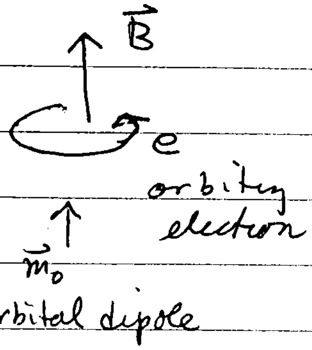
Circulating currents on atomic scale give rise to local magnetic dipole moments, which create local magnetic fields in the material.

Sources of circulating atomic currents:

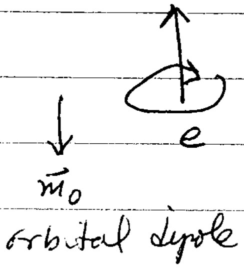
1) intrinsic angular momentum of electrons, ie "electron spin" - can add up and give a net angular momentum to atom
2) orbital angular momentum of electrons - can add up to give net angular momentum of atom.

(1) + (2) ⇒ atoms can have a <u>net</u> magnetic dipole moment. When $\vec{B}=0$, these atomic moments are generally in random orientations and average to zero (exception is a <u>ferromagnet</u> where moments can allign even if $\vec{B}=0$) When apply $\vec{B}\neq 0$, the moments tend to allign parallel to $\vec{B}$ giving a net magnetization density $\vec{M} \propto \vec{B}$. This is a <u>paramagnetic</u> effect.

But there is also a <u>diamagnetic</u> effect from orbital angular momentum (exists even if total angular momentum of electrons is zero, ie exists for atoms with zero net dipole moment)



← applying $\vec{B}$ to orbiting electron speeds up its orbital velocity. Increased angular momentum of negatively charged electron gives change in dipole moment $\Delta\vec{m} \propto -\vec{B}$



← applying $\vec{B}$ to orbiting electron slows down its orbital velocity. Net result is again that $\Delta\vec{m} \propto -\vec{B}$

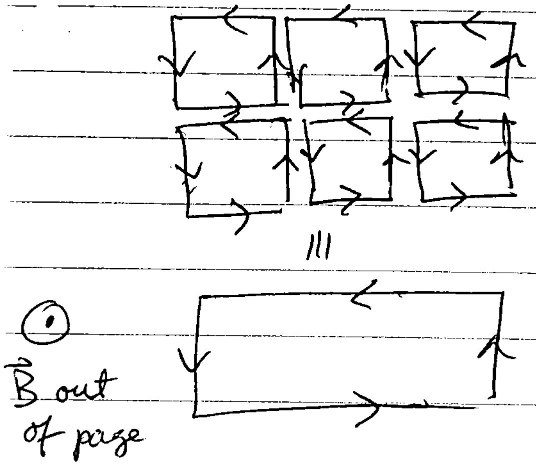
see Griffiths chpt 6 + prob 7.17 2nd ed for details

No matter which way electron orbits with respect to $\vec{B}$, result is a decrease in magnetic moment, so $\Delta\vec{m} \propto -\vec{B}$
That $\Delta\vec{m}$ is opposite to $\vec{B}$ is called diamagnetism

Model atomic magnetic moments as small current loops. When loops get oriented, ie there is non zero average magnetization density

$$\vec{M}(\vec{r}) = \sum_i \vec{m}_i\, \delta(\vec{r}-\vec{r}_i)$$

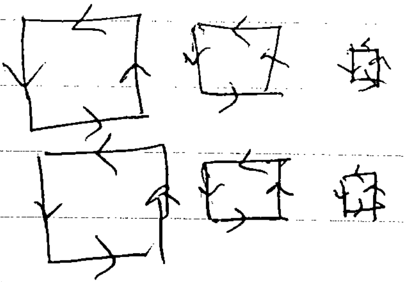
then net effect is to have a current flowing around the system. This current gives rise to magnetic fields alligned atomic moments in a uniform applied $\vec{B}$



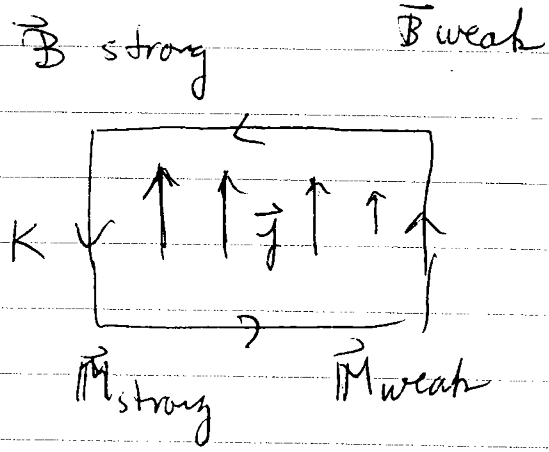
in interior, currents in opposite directions cancell and so $\vec{J} = 0$ inside

but is net circulation of current around boundary of material
⇒ surface current $\vec{K}_{bound}$

If $\vec{B}$ is not uniform, then $\vec{M}$ is not uniform
Can create finite current density $\vec{J}$ in interior, as well as surface currents



Now currents in interior do not cancell. Net current $\vec{J}_{bound}$ in interior



$\vec{B}$ out of page ⇒ $\vec{M}$ out of page
~~M varies along page~~
$\vec{M}$ varies in direction ⊥ direction of $\vec{M}$
⇒ $\vec{\nabla}\times\vec{M} \neq 0$ gives $\vec{J}_{bound}$

Average current

$$\langle \vec{j}_0 \rangle = \Big\langle \sum_{i \in \text{free}} q_i \vec{v}_i\, \delta(\vec{r}-\vec{r}_i) \Big\rangle + \sum_{\substack{n \\ \text{molec}}} \langle \vec{j}_n \rangle$$

$\uparrow$ current from free charges

$\uparrow$ current from molecule n of the dielectric

$$\langle \vec{j}_n(\vec{r},t) \rangle = \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \langle \delta(\vec{r}-\vec{r}_n(t)-\vec{r}_{ni}(t)) \rangle$$

$$= \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni})\, f(\vec{r}-\vec{r}_n(t)-\vec{r}_{ni}(t))$$

$\uparrow$ $\vec{v}_n = \dfrac{d\vec{r}_n}{dt}$ $\quad$ $\uparrow$ $\vec{v}_{ni} = \dfrac{d\vec{r}_{ni}}{dt}$ $\quad$ $\uparrow$ position of CM of molec n $\quad$ $\uparrow$ position charge i wrt CM

as with $\langle \rho_0 \rangle$, we can expand in $\vec{r}_{ni}$

$$\langle \vec{j}_n \rangle = \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \Big\{ f(\vec{r}-\vec{r}_n) - \vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)$$
$$+ \frac{1}{2} \sum_{\alpha\beta} (r_{ni})_\alpha (r_{ni})_\beta \frac{\partial^2 f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta}$$
$$+ \cdots \Big\}$$

we will keep only the first two terms in the expansion

The various terms we have to consider are

① $\sum_{i\in n} q_i \vec{v}_n f(\vec{r}-\vec{r}_n)$

② $\sum_{i\in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n)$

③ $-\sum_{i\in n} q_i \vec{v}_n \left[\vec{r}_{ni}\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n)\right]$

④ $-\sum_{i\in n} q_i \vec{v}_{ni} \left[\vec{r}_{ni}\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n)\right]$

① $= \vec{v}_n f(\vec{r}-\vec{r}_n) \sum_{i\in n} q_i = q_n \vec{v}_n f(r-r_n)$

$= \langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n)\rangle$

this is just current of molecule as if it were a point charge q_n. For a neutral molecule $q_n=0$ and this term vanishes.

② Note: $\frac{\partial}{\partial t}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle = \frac{\partial}{\partial t}\Big(\sum_{i\in n} q_i \vec{r}_{ni} f(\vec{r}-\vec{r}_n$

$$= \sum_{i\in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n)$$

$$+ \sum_{i\in n} q_i \vec{r}_{ni}\left[-\vec{\nabla} f(\vec{r}-\vec{r}_n)\cdot\vec{v}_n\right]$$

So for ②, $\sum_{i\in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n)$

$$= \frac{\partial}{\partial t}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$$

$$+\left[\vec{v}_n\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n)\right]\vec{p}_n$$

So

$$② = \sum_{i\in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n) = \frac{\partial}{\partial t}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle + (\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(r-r_n)\rangle$$

2nd term is $\sum_\alpha v_{n\alpha}\frac{\partial}{\partial r_\alpha}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$

$$③ = -\vec{v}_n \Big(\sum_{i\in n} q_i \vec{r}_{ni}\Big)\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n) = -\vec{v}_n\left(\vec{p}_n\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n)\right)$$

$$= -\vec{v}_n\, \vec{\nabla}\cdot\langle \vec{p}_n\delta(\vec{r}-\vec{r}_n)\rangle = \sum_\alpha \vec{v}_n \frac{\partial}{\partial r_\alpha}\langle p_{n\alpha}\delta(\vec{r}-\vec{r}_n)\rangle$$

$$④ = -\vec{\nabla} f(\vec{r}-\vec{r}_n)\cdot\sum_{i\in n} q_i \vec{r}_{ni}\vec{v}_{ni}$$

We have seen the tensor $\sum_i q_i \vec{r}_{ni}\vec{v}_{ni}$ before when we considered the magnetic dipole moment

$$\sum_{i\in n} q_i \vec{r}_{ni}\vec{v}_{ni} = \int d^3r\, \vec{r}\,\vec{j}$$

where $\vec{j}(\vec{r}) \equiv \sum_{i\in n} q_i\vec{v}_{ni}\delta(\vec{r}-\vec{r}_{n}$ is current density with respect to center of mass of molecule

We had $\int d^3r\,\vec{r}\,\vec{j} = -\int d^3r\,\vec{j}\,\vec{r} - \int d^3r\,(\vec{\nabla}\cdot\vec{j})\,\vec{r}\,\vec{r}$

in statics, $\vec{\nabla}\cdot\vec{j} = 0$

in general $\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$

$$\int d^3r\,\vec{r}\,\vec{j} = -\int d^3r\,\vec{j}\,\vec{r} + \int d^3r\,\frac{\partial\rho}{\partial t}\,\vec{r}\,\vec{r}$$

$$= -\int d^3r\,\vec{j}\,\vec{r} + \frac{\partial}{\partial t}\left[\int d^3r\,\rho\,\vec{r}\,\vec{r}\right]$$

although this is not zero, it is a <u>quadrapole</u> term of the same order as the term we dropped when we truncate expansion to linear order

$$\sim O\left(\frac{a_0}{L}\right)^2$$

So $\int d^3r \, \vec{r}\,\vec{j} \simeq -\int d^3r \, \vec{j}\,\vec{r}$ ignoring the quadrupole term

$$= \frac{1}{2}\int d^3r \left[\vec{r}\,\vec{j} - \vec{j}\,\vec{r} \right]$$

$$\sum_{i\in n} q_i \vec{r}_{ni}\,\vec{v}_{ni} = \frac{1}{2}\sum_{i\in n} q_i \left[\vec{r}_{ni}\,\vec{v}_{ni} - \vec{v}_{ni}\,\vec{r}_{ni} \right]$$

$$-\vec{\nabla} f(r-r_n) \cdot \sum_{i\in n} q_i \vec{r}_{ni}\,\vec{v}_{ni} = -\vec{\nabla} f(\vec{r}-\vec{r}_n) \cdot \frac{1}{2}\sum_{i} \big[\vec{r}_{ni}\,\vec{v}_{ni} - \vec{v}_{ni}\,\vec{r}_{n}$$

$$= \frac{-1}{2}\sum_{i\in n} q_i \left[(\vec{\nabla} f \cdot \vec{r}_{ni})\vec{v}_{ni} - (\vec{\nabla} f \cdot \vec{v}_{ni})\vec{r}_{ni} \right]$$

$$= -\frac{1}{2}\sum_{i\in n} q_i \, \vec{\nabla} f \times (\vec{v}_{ni} \times \vec{r}_{ni})$$ triple product rule

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n) \times \frac{1}{2}\sum_{i\in n} \vec{r}_{ni} \times \vec{v}_{ni}\, q_i$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n) \times \frac{1}{2}\int d^3r \, \vec{r} \times \vec{j}$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n) \times c\,\vec{m}_n$$ where $\vec{m}_n = \frac{1}{2c}\sum_{i\in n} \bar{r}_{ni} \times \vec{v}_{ni}\, q_i$

is magnetic dipole moment of molecule n

$$= \vec{\nabla} \times f(\vec{r}-\vec{r}_n)\, c\,\vec{m}_n$$

$$= \vec{\nabla} \times \left\langle c\,\vec{m}_n\, \delta(\vec{r}-\vec{r}_n) \right\rangle$$

Adding all the pieces

$$\langle \vec{j}_n \rangle = \underset{①}{\langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n) \rangle} + c\vec{\nabla} \times \underset{④}{\langle \vec{m}_n \delta(\vec{r}-\vec{r}_n) \rangle}$$

$$+ \underset{②}{\frac{\partial}{\partial t} \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle} + \underset{②}{(\vec{v}_n \cdot \vec{\nabla}) \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle}$$

$$- \underset{③}{\vec{v}_n \vec{\nabla} \cdot \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle}$$

Define $\vec{\mathbb{M}}(\vec{r}) \equiv \sum_n \langle \vec{m}_n \delta(\vec{r}-\vec{r}_n) \rangle$ average magnetization density

$\vec{\mathbb{P}}(\vec{r}) \equiv \sum_n \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle$ polarization density, as before

$$\sum_n \langle \vec{j}_n \rangle = \sum_n \langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n) \rangle + c\vec{\nabla} \times \vec{\mathbb{M}} + \frac{\partial \vec{\mathbb{P}}}{\partial t}$$

$$+ \sum_n \left[(\vec{v}_n \cdot \vec{\nabla}) \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle - \vec{v}_n \vec{\nabla} \cdot \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle \right]$$

see Jackson (6.96) for additional electric quadrapole terms

The last term on the right hand side is usually small and ignored. This is because the molecular velocities $\vec{v}_n$ are usually small, and randomly oriented, so that they average to zero. (see Jackson (6.100) for case of net translation of dielectric, $\vec{v}_n$ = const all n)

Define macroscopic current density

$$\vec{j}(\vec{r},t) = \Big\langle \sum_{i\in \text{free}} q_i \underset{\uparrow}{\vec{v}_i}\, \delta(\vec{r}-\vec{r}_i) \Big\rangle + \Big\langle \sum_{\substack{n \\ \text{molecs}}} q_n \underset{\uparrow}{\vec{v}_n}\, \delta(\vec{r}-\vec{r}_n) \Big\rangle$$

current of free charges

current of molecular drifting if molecules are charged

Then $\langle \vec{j}_0 \rangle = \vec{j} + c\, \vec{\nabla}\times\vec{\mathbb{M}} + \dfrac{\partial \vec{\mathbb{P}}}{\partial t}$

Ampere's law becomes upon averaging

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\langle \vec{j}_0 \rangle + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$= \frac{4\pi}{c}\vec{j} + 4\pi\vec{\nabla}\times\vec{\mathbb{M}} + \frac{4\pi}{c}\frac{\partial \vec{\mathbb{P}}}{\partial t} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\left(\vec{B} - 4\pi\vec{\mathbb{M}}\right) = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E} + 4\pi\vec{\mathbb{P}}\right)$$

define $\boxed{\vec{H} \equiv B - 4\pi\vec{\mathbb{M}}}$ to get

$$\boxed{\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}} \qquad \boxed{\vec{D} = \vec{E} + 4\pi\vec{\mathbb{P}}} \text{ as before}$$

official nomenclature: $\vec{B}$ is the magnetic <u>induction</u>
$\vec{H}$ is the magnetic <u>field</u>

common usage: both $\vec{H}$ and $\vec{B}$ are called magnetic field