When atoms have intrinsic magnetic moments due to electron spin, we can add these to $\vec{M}$ in obvious way

When molecules are neutral, $q_n = 0$, the "bound current" is given by

$$\vec{J}_{bound} = \sum_n \langle \vec{J}_n \rangle = c\,\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Note that the $\frac{\partial \vec{P}}{\partial t}$ term is crucial to give conservation of bound charge

$$\vec{\nabla}\cdot\vec{J}_{bound} = c\,\vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) + \vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t}$$

$$= 0 + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P})$$

$$= \frac{-\partial \rho_{bound}}{\partial t}$$ where $\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$ is bound charge density

So $$\boxed{\vec{\nabla}\cdot\vec{J}_{bound} + \frac{\partial \rho_{bound}}{\partial t} = 0}$$

and bound charge is conserved.

Since total average charge must be conserved, ie

$$\vec{\nabla}\cdot\langle\vec{J}_0\rangle - \frac{\partial\langle\rho_0\rangle}{\partial t} = 0,$$ and $\langle\vec{J}_0\rangle = \vec{J} + \vec{J}_{bound}$

free current

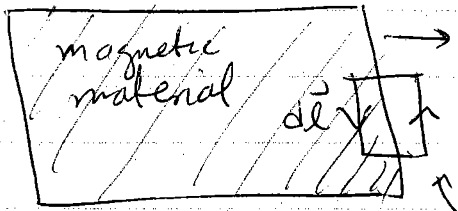
$\langle\rho_0\rangle = \rho + \rho_{bound}$

free charge

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

free charge is also conserved

At a surface of a magnetic material



$\to \hat{n}$ outward normal to surface

take $\hat{z} \equiv d\vec{l} \times \hat{n}$ out of page

Amperean loop C bounding surface S of area da

$$c \int_S da\, \hat{z} \cdot (\vec{\nabla} \times \vec{M}) = \int_S da\, \hat{z} \cdot \vec{J}_{bound} = da\, \hat{z} \cdot \vec{J}_{bound}$$

$$= (d\vec{l} \times \hat{n}) \cdot \vec{K}_{bound} \quad \text{as width of loop} \to 0$$

$$= (\hat{n} \times \vec{K}_{bound}) \cdot d\vec{l}$$

But by Stokes theorem

$$c \int_S da\, \hat{z} \cdot (\vec{\nabla} \times \vec{M}) = c \int_C d\vec{l} \cdot \vec{M} = c\, d\vec{l} \cdot \vec{M}$$

since width $\to 0$ and $\vec{M} = 0$ outside

$$\Rightarrow \quad c\, d\vec{l} \cdot \vec{M} = (\hat{n} \times \vec{K}_{bound}) \cdot d\vec{l}$$

for any $d\vec{l}$ in plane of surface

$$\Rightarrow \quad c\vec{M}_t = \hat{n} \times \vec{K}_{bound}$$

where $\vec{M}_t$ is component of $\vec{M}$ tangential to the surface (since $\vec{K}_b$ is in plane of surface, $\hat{n} \times \vec{K}$ is also entirely in the plane of the surface)

$$\Rightarrow \quad c\, \hat{n} \times \vec{M}_t = c\, \hat{n} \times \vec{M} = \hat{n} \times (\hat{n} \times \vec{K}_{bound})$$

$$= -\vec{K}_{bound}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \vec{K}_{bound} &= c\, \vec{M} \times \hat{n} \\ \vec{J}_{bound} &= c\, \vec{\nabla} \times \vec{M} \end{aligned}}$$

Total bound charge vanishes

$$Q_{bound} = \int_V d^3r\ \rho_{bound} + \int_S da\ \sigma_{bound}$$

vol of dielectric   surface of dielectric

$$= \int_V d^3r\ -\vec{\nabla}\cdot\vec{P} + \int da\ \hat{n}\cdot\vec{P}$$

but by Gauss theorem $\int_V d^3r\ \vec{\nabla}\cdot\vec{P} = \int da\ \hat{n}\cdot\vec{P}$

so $Q_{bound} = -\int da\ \hat{n}\cdot\vec{P} + \int da\ \hat{n}\cdot\vec{P} = 0$

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \qquad \vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

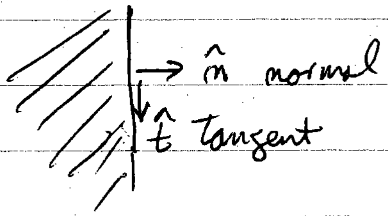
where ρ and $\vec{j}$ are macroscopic charge + current densities
do not include bound charges or currents

$\vec{D} = \vec{E} + 4\pi\vec{P}$, $\vec{P}$ is polarization density

$\vec{H} = \vec{B} - 4\pi\vec{M}$, $\vec{M}$ is magnetization density

Boundary conditions for statics

electrostatics: at surface of a dielectric, or at interface between two different dielectrics



$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \hat{t}\cdot\vec{E}_{above} = \hat{t}\cdot\vec{E}_{below}$

tangential component $\vec{E}$ is continuous

$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \Rightarrow \hat{n}\cdot(\vec{D}_{above} - \vec{D}_{below}) = 4\pi\sigma$

normal component of $\vec{D}$ jumps by $4\pi\sigma$

magnetostatics: at surface or interface of magnetic materials

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \hat{n}\cdot\vec{B}_{above} - \hat{n}\cdot\vec{B}_{below}$

normal component of $\vec{B}$ is continuous

$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} \Rightarrow \hat{t}\cdot(\vec{H}_{above} - \vec{H}_{below}) = \frac{4\pi}{c}\vec{K}$

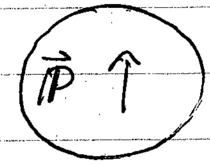
tangential component of $\vec{H}$ jumps by $\frac{4\pi}{c}\vec{K}$

if $\sigma = 0$, ie no free surface charge, then $\hat{n}\cdot\vec{D}$ continuous

if $\vec{K} = 0$, ie no free surface current, then $\hat{t}\cdot\vec{H}$ continuous

Examples

① <u>Uniformly polarized sphere</u> of radius R $\qquad \vec{\mathbb{P}} = \mathbb{P}\hat{z}$



bound charge $\quad \rho_b = -\vec{\nabla}\cdot\vec{\mathbb{P}} = 0$ as $\vec{\mathbb{P}}$ constant

$$\sigma_b = \hat{n}\cdot\vec{\mathbb{P}} = \hat{r}\cdot\vec{\mathbb{P}} = \mathbb{P}\cos\theta$$

we saw earlier that a sphere with surface charge $\sigma(\theta) = \sigma_0 \cos\theta$ gives an electric field like a pure dipole for $r > R$, and is constant for $r < R$.

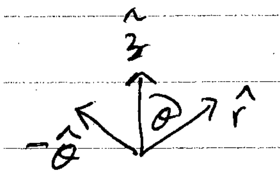
$$\vec{E}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 \mathbb{P}\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r > R \\ -\dfrac{4\pi\mathbb{P}}{3}\,\hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{p} = \frac{4}{3}\pi R^3 \vec{\mathbb{P}}$

<u>check behavior at boundary</u>

<u>Tangential component $\vec{E}$</u>

$$\vec{E}^{\,t}_{above} = \left(\tfrac{4}{3}\pi R^3 \mathbb{P}\right)\frac{\sin\theta\,\hat{\theta}}{R^3} = \tfrac{4}{3}\pi\mathbb{P}\sin\theta\,\hat{\theta}$$



$$\vec{E}^{\,t}_{below} = -\frac{4\pi\mathbb{P}}{3}(\hat{z}\cdot\hat{\theta})\,\hat{\theta} = \tfrac{4}{3}\pi\mathbb{P}\sin\theta\,\hat{\theta}$$

$\Rightarrow$ Tangential component $\vec{E}$ is continuous

<u>normal component</u> of $\vec{D}$

outside: $\vec{\mathbb{P}} = 0 \Rightarrow \vec{D} = \vec{E}$

$$\Rightarrow \hat{n}\cdot\vec{D} = \hat{r}\cdot\vec{E} = \left(\tfrac{4}{3}\pi R^3\mathbb{P}\right)\frac{2\cos\theta}{R^3} = \frac{8}{3}\pi\mathbb{P}\cos\theta$$

inside: $\vec{E} = -\frac{4\pi\vec{P}}{3} \Rightarrow \vec{P} = -\frac{3}{4\pi}\vec{E}$

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \vec{E} - 3\vec{E} = -2\vec{E} = \frac{8\pi}{3}P\,\hat{z}$$

$$\hat{n}\cdot\vec{D} = \hat{r}\cdot\left(\frac{8\pi}{3}P\hat{z}\right) = \frac{8\pi}{3}P\cos\theta$$

$\Rightarrow$ normal component $\vec{D}$ is continuous

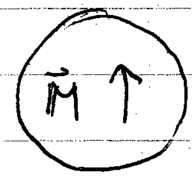
Note: normal component of $\vec{E}$ should jump by $4\pi\sigma_b = 4\pi P\cos\theta$

to check this: $\hat{n}\cdot\vec{E}^{\text{inside}} = \hat{r}\cdot\left(-\frac{4}{3}\pi P\hat{z}\right) = -\frac{4}{3}\pi P\cos\theta$

$$\hat{n}\cdot\left(\vec{E}^{\text{above}} - \vec{E}^{\text{below}}\right) = \frac{8}{3}\pi P\cos\theta + \frac{4}{3}\pi P\cos\theta$$

$$= \frac{12}{3}\pi P\cos\theta = 4\pi P\cos\theta = 4\pi\sigma_b(\theta)$$

② Uniformly magnetized sphere of radius R $\quad \vec{M} = M\hat{z}$



bound current $\quad \vec{J}_b = c\,\vec{\nabla}\times\vec{M} = 0$ as $\vec{M}$ constant

$$\vec{K}_b = c\,\vec{M}\times\hat{n} = cM\,(\hat{z}\times\hat{r})$$
$$= cM\sin\theta\,\hat{\varphi}$$

we saw earlier that a sphere with surface current $\vec{K}_b = K_0 \sin\theta\,\hat{\theta}$ gives a magnetic field that is pure dipole for $r > R$, and is constant for $r < R$.

$$\vec{B}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 M\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r > R \\ \frac{8}{3}\pi M\,\hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$

check behavior at boundary

normal component of $\vec{B}$

$$\hat{n} \cdot \vec{B}^{\,above} = \hat{r} \cdot \vec{B}^{\,above} = \frac{8}{3}\pi M \cos\theta$$

$$\hat{n} \cdot \vec{B}^{\,below} = \hat{r} \cdot \vec{B}^{\,below} = \frac{8}{3}\pi M(\hat{r}\cdot\hat{z}) = \frac{8}{3}\pi M \cos\theta$$

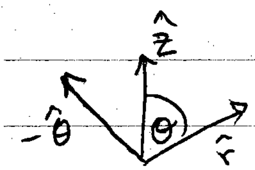
$\Rightarrow$ normal component of $\vec{B}$ is continuous

tangential component of $\vec{H}$

outside: $\vec{M}=0 \Rightarrow \vec{H}=\vec{B}$

$$\vec{H}^{t}_{above} = \left(\tfrac{4}{3}\pi M\right)\sin\theta\,\hat{\theta}$$

inside:
$$\vec{H} = \vec{B} - 4\pi\vec{M} = \vec{B} - 4\pi\left(\frac{3}{8\pi}\vec{B}\right) = \vec{B} - \frac{3}{2}\vec{B} = -\frac{1}{2}\vec{B}$$
$$= -\frac{4\pi}{3}M\hat{z}$$



so
$$\vec{H}^{t}_{below} = -\frac{4\pi}{3}M(\hat{z}\cdot\hat{\theta}) = \frac{4\pi}{3}M\sin\theta\,\hat{\theta}$$

$\Rightarrow$ tangential component $\vec{H}$ is continuous

Note: tangential component $\vec{B}$ should jump by $\frac{4\pi}{c}\vec{K}_b = 4\pi M\sin\theta\,\hat{\theta}$

inside:

to check:
$$\vec{B}^{t}_{below} = \frac{8}{3}\pi M(\hat{z}\cdot\hat{\theta})\hat{\theta} = -\frac{8}{3}\pi M\sin\theta\,\hat{\theta}$$

$$\vec{H}^{t}_{above} = \vec{B}^{t}_{above} \Rightarrow \vec{B}^{t}_{above} - \vec{B}^{t}_{below} = \frac{4}{3}\pi M\sin\theta\,\hat{\theta} + \frac{8}{3}\pi M\sin\theta\,\hat{\theta}$$

$$= 4\pi M\sin\theta\,\hat{\theta} = \frac{4\pi}{c}\vec{K}_b$$

Linear Materials

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

where ρ and $\vec{j}$ are macroscopic charge + current densities and

$\vec{D} = \vec{E} + 4\pi\vec{P}$ $\quad$ $\vec{P}$ is polarization density

$\vec{H} = \vec{B} - 4\pi\vec{M}$ $\quad$ $\vec{M}$ is magnetization density

To close these equations, we will in general need to know how $\vec{P}$ and $\vec{M}$ are related to the $\vec{E}$ and $\vec{B}$ in the material.

In some materials, there can be a finite $\vec{P}$ or $\vec{M}$ even if $\vec{E}$ and $\vec{B}$ are zero:

Ferromagnet: $\vec{M}$ can be non zero even if $\vec{B} = 0$

Ferroelectric: $\vec{P}$ can be non zero even if $\vec{E} = 0$

But more common are linear materials in which, for small $\vec{E}$ and $\vec{B}$, one has $\vec{P} \propto \vec{E}$ and $\vec{M} \propto \vec{B}$.

linear dielectric

$\vec{P} = \chi_e \vec{E}$ $\qquad$ χ_e is "electric susceptibility"

$\chi_e > 0$ for statics

$\vec{D} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi \chi_e) \vec{E}$

$\vec{D} = \epsilon \vec{E}$ $\qquad$ with $\epsilon = 1 + 4\pi \chi_e$

ϵ is the dielectric constant

linear magnetic materials

$\vec{M} = \chi_m \vec{H}$ $\qquad$ χ_m is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi \chi_m \vec{H}$

$\vec{B} = (1 + 4\pi \chi_m) \vec{H}$

$\vec{B} = \mu \vec{H}$ $\qquad$ with $\mu = 1 + 4\pi \chi_m$

μ is magnetic permeability

For statics, $\chi_e > 0$ and χ_m (or alternatively ϵ and μ) are constants depending on the material.

When we consider dynamics we will see that ϵ becomes a function of frequency.