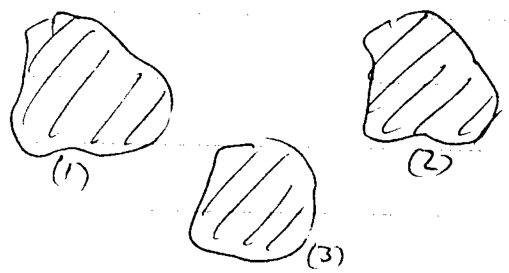
Capacitance

Consider a set of conductors with potential $\phi(\vec{r}) = V_i$ fixed on conductor i



(also need condition on $V(\vec{r}) \to \infty$ if system is not enclosed)

From uniqueness theorem we know that specifying the V_i on each conductor is enough to determine the potential $\phi(\vec{r})$ everywhere. We can write this potential in the following form.

Let $\phi^{(i)}(\vec{r})$ be the solution to the boundary value problem $\nabla^2\phi^{(i)}(r) = 0$ and $\phi^{(i)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor } (i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor } (j),\ j \neq i \end{cases}$

Then by superposition

$$\phi(\vec{r}) = \sum_i V_i \, \phi^{(i)}(\vec{r})$$

is solution to the problem $\nabla^2\phi = 0$ and $\phi(\vec{r}) = V_i$ for $\vec{r}$ on surface of conductor (i)

The surface charge density at $\vec{r}$ on surface of conductor (i) is

$$\sigma^{(i)}(\vec{r}) = \frac{-1}{4\pi}\frac{\partial\phi(\vec{r})}{\partial n} = -\frac{1}{4\pi}\sum_j V_j \frac{\partial\phi^{(j)}(\vec{r})}{\partial n}$$

where $\frac{\partial\phi}{\partial n} = (\vec{\nabla}\phi)\cdot\hat{n}$ is the derivative normal to the surface at point $\vec{r}$.

The total charge on conductor (i) is

$$Q_i = \int_{S_i} da\, \sigma^{(i)}(\vec{r}) = -\frac{1}{4\pi} \sum_j V_j \int_{S_i} da \frac{\partial \phi^{(j)}}{\partial n}$$

surface of conductor(i)

Define $C_{ij} \equiv -\frac{1}{4\pi} \int_{S_i} da \frac{\partial \phi^{(j)}}{\partial n}$

the C_{ij} depend only on the geometry of the conductors

Then we have

$$\boxed{Q_i = \sum_j C_{ij} V_j}$$

C_{ij} is the capacitance matrix

The charge on conductor (i) is a linear function of the potentials V_j on the conductors (j)

Since we know that specifying the Q_i that is on each conductor will uniquely determine $\phi(\vec{r})$ and hence the potential V_i on each conductor, the capacitance matrix is invertable

$$V_i = \sum_j [C^{-1}]_{ij} Q_j$$

The electrostatic energy of the conductors is then

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\, \phi = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_{i,j} C_{ij} V_i V_j$$

Common to define capacitance of <u>two</u> conductors by

$$C = \frac{Q}{V_1 - V_2}$$

when conductor (1) has charge Q
conductor (2) has charge $-Q$
$V_1 - V_2$ is potential difference between the two conductors.
all other conductors fixed at $V_i = 0$

We can determine C in terms of the elements of the matrix C_{ij}

$$\left.\begin{aligned} Q &= C_{11} V_1 + C_{12} V_2 \\ -Q &= C_{21} V_1 + C_{22} V_2 \end{aligned}\right\} \Rightarrow V_2 = -\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right) V_1$$

$$\Rightarrow Q = \left[C_{11} - C_{12} \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1$$

$$V_1 - V_2 = \left[1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1$$

$$C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12}\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)}{1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)}$$

$$\boxed{C = \frac{C_{11} C_{22} - C_{12} C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}}$$

Capacitance can also be defined when the space between the conductors is filled with a dielectric ϵ. In this case, if Q_i is the free charge, then Q_i/ϵ is the effective total charge to use in computing ϕ.

$$\Rightarrow \frac{Q_i}{\epsilon} = \sum_j C_{ij}^{(0)} V_j$$ where $C_{ij}^{(0)}$ are capacitances appropriate to a vacuum between the conductors

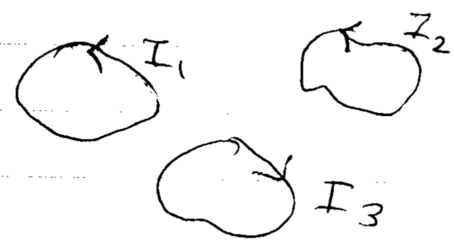
$$\Rightarrow Q_i = \sum_j \epsilon\, C_{ij}^{(0)} V_j$$

$$= \sum_j C_{ij} V_j \quad \text{where } C_{ij} = \epsilon\, C_{ij}^{(0)}$$

the capacitance is increased by a factor the dielectric constant ϵ.

Inductance

Consider a set of current carrying loops C_i with currents I_i



In Coulomb gauge, we can write the magnetic vector potential $\vec{A}$ from these current loops as

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \sum_i \frac{I_i}{c} \oint_{C_i} \frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

↑ integrate over loop C_i, integration variable is $\vec{r}'$

The magnetic flux through loop i is

$$\Phi_i = \int_{S_i} da\, \hat{n}\cdot\vec{B} = \int_{S_i} da\, \hat{n}\cdot\vec{\nabla}\times\vec{A} = \oint_{C_i} d\vec{l}\cdot\vec{A}$$

↑ surface bounded by loop C_i

$$\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i}\oint_{C_j} \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

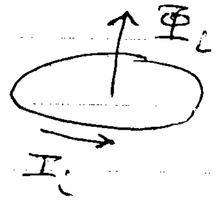
$$\boxed{\Phi_i \equiv c\sum_j M_{ij} I_j}$$

where $M_{ij} = \oint_i \oint_j \frac{d\vec{l}_i\cdot d\vec{l}_j}{c^2|\vec{r}-\vec{r}'|}$ ← pure geometrical quantity

is the <u>mutual inductance</u> of loops (i) and (j). $M_{ij} = M_{ji}$

$L_i \equiv M_{ii}$ is <u>self-inductance</u> of loop (i)

The sign convention in the above is that Φ_i is computed in direction given by right hand rule, according to the direction taken for current in loop (i)



Magnetostatic energy

$$\mathcal{E} = \frac{1}{2c}\int d^3r\, \vec{j}\cdot\vec{A} = \frac{1}{2c}\sum_i \oint_{C_i} d\vec{l}\cdot\vec{A}\, I_i$$

$$= \frac{1}{2c}\sum_i \Phi_i I_i$$

$$\mathcal{E} = \frac{1}{2}\sum_{i,j} M_{ij} I_i I_j$$

Force and torque on electric dipoles

localized charge distribution $\rho(\vec{r})$ with net charge $\underline{\int d^3r\, \rho = 0}$

force on ρ in slowly varying electric field $\vec{E}$ is

$$\vec{F} = \int d^3r\, \rho(\vec{r})\, \vec{E}(\vec{r})$$

define $\vec{r} = \vec{r}_0 + \vec{r}\,'$ where $\vec{r}_0$ is some fixed reference point in center of charge distr ρ, and $\vec{r}\,'$ is distance relative to $\vec{r}_0$

$$\vec{F} = \int d^3r'\, \rho(\vec{r}\,')\, \vec{E}(\vec{r}_0 + \vec{r}\,')$$

since $\vec{E}$ is slowly varying on length scale where $\rho \neq 0$, we expand

$$\vec{F} \cong \int d^3r'\, \rho(\vec{r}\,') \left[\vec{E}(\vec{r}_0) + (\vec{r}\,' \cdot \vec{\nabla}) \vec{E}(\vec{r}_0) \right] + \cdots$$

$$= \vec{E}(\vec{r}_0) \int d^3r'\, \rho(\vec{r}\,') + \left(\int d^3r'\, \rho(r')\, \vec{r}\,' \cdot \vec{\nabla} \right) \vec{E}(\vec{r}_0)$$

$$= 0 + (\vec{p} \cdot \vec{\nabla}) \vec{E}(\vec{r}_0)$$

$$\boxed{\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} = \sum_{\alpha=1}^{3} p_\alpha \frac{\partial \vec{E}}{\partial r_\alpha}}$$

For $\vec{E}$ = constant, $\vec{F} = 0$

Torque on ρ is ~~[illegible]~~

$$\vec{N} = \int d^3r\, \rho(\vec{r})\, \vec{r} \times \vec{E}(\vec{r}) \cong \int d^3r\, \rho(\vec{r})\, \vec{r} \times \left[\vec{E}(\vec{r}_0) + \cdots\right]$$

to lowest order $\boxed{\vec{N} = \vec{p} \times \vec{E}}$

<u>Force and torque on magnetic dipoles</u>

localized <u>magnetostatic</u> current distribution $\vec{j}(\vec{r})$

$$\vec{F} = \frac{1}{c}\int d^3r\, \vec{j} \times \vec{B}$$

expand about center of current $\vec{r}_0$

$$\vec{B}(\vec{r}) \simeq \vec{B}(\vec{r}_0) + (\vec{r}' \cdot \vec{\nabla})\vec{B}(\vec{r}_0) + \cdots$$

$$\vec{F} = \frac{1}{c}\left[\int d^3r'\, \vec{j}(\vec{r}')\right] \times \vec{B}(\vec{r}_0) + \frac{1}{c}\int d^3r'\, \vec{j}(\vec{r}') \times (\vec{r}' \cdot \vec{\nabla})\vec{B}(\vec{r}_0)$$

from discussion of magnetic dipole approx we had $\int d^3r\, \vec{j} = 0$ for magnetostatics where $\vec{\nabla} \cdot \vec{j} = 0$. So 1st term vanishes. The 2nd term can be written as

$$F_\alpha = \frac{\varepsilon_{\alpha\beta\gamma}}{c}\int d^3r'\, j_\beta\, r'_\delta\, \partial_\delta B_\gamma$$

for magnetostatics see magnetic dipole derivation

we need the tensor $\frac{1}{c}\int d^3r'\, j_\beta\, r'_\delta = -\frac{1}{c}\int d^3r'\, r'_\beta\, j_\delta$

$$= \frac{1}{2c}\int d^3r'\left[j_\beta r'_\delta - r'_\beta j_\delta\right]$$

$$= -m_\sigma\, \varepsilon_{\sigma\beta\delta}$$

magnetic dipole $\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r} \times \vec{j}$

$$F_\alpha = \varepsilon_{\alpha\beta\gamma}\, \varepsilon_{\sigma\beta\delta}(-m_\sigma)\, \partial_\delta B_\gamma$$

$$= -\left(\delta_{\alpha\sigma}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\sigma\gamma}\right) m_\sigma \partial_\delta B_\gamma$$

$$= \underset{\alpha}{\vec{\nabla}}(\vec{m}\cdot\vec{B}) - \vec{m}_\alpha \vec{\nabla}\cdot\vec{B}$$

$$\boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})} \qquad \text{as } \vec{\nabla}\cdot\vec{B} = 0$$

torque on $\vec{j}$ is

$$\vec{N} = \frac{1}{c}\int d^3r\; \vec{r}\times(\vec{j}\times\vec{B})$$

to lowest order, $\vec{B} = \vec{B}(\vec{r}_0)$ is const over region where $\vec{j} \neq 0$

$$= \frac{1}{c}\int d^3r \left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{B}\,(\vec{r}\cdot\vec{j})\right]$$

2nd term $= 0$ as follows

$$\int d^3r\; \vec{r}\cdot\vec{j} = \int d^3r\; \vec{j}\cdot\vec{\nabla}\left(\frac{r^2}{2}\right) \qquad \text{as } \vec{\nabla}\left(\frac{r^2}{2}\right) = \vec{r}$$

$$= -\int d^3r\,(\vec{\nabla}\cdot\vec{j})\left(\frac{r^2}{2}\right)$$

integrate by parts. Surface term $\to 0$ as $\vec{j}$ is localized

$$= 0 \qquad \text{as } \vec{\nabla}\cdot\vec{j} = 0 \text{ in magnetostatics}$$

1st term involves

see derivation of magnetic dipole approx

$$\int d^3r\; \vec{j}\,\vec{r} = -\int d^3r\; \vec{r}\,\vec{j} = \frac{1}{2}\int d^3r\left[\vec{j}\,\vec{r} - \vec{r}\,\vec{j}\right]$$

So

$$\vec{N} = \frac{1}{2c}\int d^3r\left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{r}\,(\vec{j}\cdot\vec{B})\right]$$

$$\vec{N} = \frac{1}{2c}\int d^3r \left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{r}\,(\vec{j}\cdot\vec{B})\right]$$

$$\cancel{= \vec{m}\times\vec{B}}$$

$$= \frac{1}{2c}\int d^3r\,(\vec{r}\times\vec{j})\times\vec{B}$$

$$\boxed{\vec{N} = \vec{m}\times\vec{B}}$$