Electrostatic energy of interaction

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\; E^2$$

Suppose the charge density ρ that produces $\vec{E}$ can be broken into two pieces, $\rho = \rho_1 + \rho_2$ with $\vec{E} = \vec{E}_1 + \vec{E}_2$ where $\vec{\nabla}\cdot\vec{E}_1 = 4\pi\rho_1$ and $\vec{\nabla}\cdot\vec{E}_2 = 4\pi\rho_2$

Then

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r \left[E_1^2 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2 \right]$$

"self-energy" of ρ_1 "self energy" of ρ_2 "interaction" energy of ρ_1 with ρ_2

$$\mathcal{E}_{int} = \frac{1}{4\pi}\int d^3r\; \vec{E}_1\cdot\vec{E}_2$$

$$= \int d^3r\; \rho_1\phi_2 = \int d^3r\; \rho_2\phi_1$$

where $\vec{E}_1 = -\vec{\nabla}\phi_1$, $\vec{E}_2 = -\vec{\nabla}\phi_2$, by similar manipulations as earlier

integrals are over <u>all</u> space

Apply to the interaction energy of a dipole in an external $\vec{E}$ field

$$\mathcal{E}_{int} = \int d^3r\; \rho_1\,\phi_2$$

ϕ_2 ← potential of external $\vec{E}$ field

ρ_1 ↑ charge distribution of dipole

Assuming ϕ_2 varys slowly on length scale of ρ_1 then we can expand $\phi_2(\vec{r}) = \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0)\cdot\vec{\nabla}\phi_2(\vec{r}_0)$ where $\vec{r}_0$ is the center of mass or any other convenient reference position within ρ_1

$$\mathcal{E}_{int} = \int d^3r\, \rho_1(\vec{r})\left[\phi_2(\vec{r}_0) + (\vec{r}-\vec{r}_0)\cdot\vec{\nabla}\phi_2(\vec{r}_0)\right]$$

$$= q\,\phi_2(\vec{r}_0) + \left[\int d^3r\, \rho_1(\vec{r})(\vec{r}-\vec{r}_0)\right]\cdot\vec{\nabla}\phi_2(\vec{r}_0)$$

$$= q\,\phi_2(\vec{r}_0) - \vec{p}\cdot\vec{E}$$

where q is total charge in ρ_1 and $\vec{p}$ is dipole moment with respect to $\vec{r}_0$. $\vec{E} = -\vec{\nabla}\phi_2$ is external $\vec{E}$-field

For a <u>neutral charge distribution</u> $q = 0$, and $\vec{p}$ is independent of the origin about which it is computed, so

$$\boxed{\mathcal{E}_{int} = -\vec{p}\cdot\vec{E}}$$

← does <u>not</u> include the energy needed to make the dipole or to make $\vec{E}$.

$\mathcal{E}_{int}$ is lowest when $\vec{p} \parallel \vec{E}$

⇒ in thermal ensemble, dipoles tend to allign parallel to an applied $\vec{E}$.

Energy of magnetic dipole in external field

We had that the force on the dipole was

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

if we regard the force as comming from the gradient of a potential energy U then $\vec{F} = -\vec{\nabla} U \Rightarrow$

$$\boxed{U = -\vec{m} \cdot \vec{B}}$$

or equivalently, energy = work done to move dipole into position from ∞

$$W = -\int_{\infty}^{\vec{r}} \vec{F} \cdot d\vec{l} = -\int_{\infty}^{\vec{r}} \vec{\nabla}(m \cdot \vec{B}) \cdot d\vec{l} = -\vec{m} \cdot \vec{B}(\vec{r})$$

This is the correct energy to use in cases where $\vec{m}$ is due to intrinsic magnetic moments of atom or molecule - say from electron or nuclear spin. For a thermal ensemble magnetic moments tend to allign $\parallel$ to $\vec{B}$.

The answer comes out quite differently if we are talking about a magnetic moment produced by a classical current loop. To see this, consider what we would get if we tried to do the calculation in a similar way to how we did if the the energy of an electric dipole in an electric field...

Magnetostatic energy of interaction

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\ B^2$$

Suppose current $\vec{j}$ that produces $\vec{B}$ can be divided $\vec{j} = \vec{j}_1 + \vec{j}_2$ with $\vec{B} = \vec{B}_1 + \vec{B}_2$ where $\vec{\nabla}\times\vec{B}_1 = \frac{4\pi}{c}\vec{j}_1$ and $\vec{\nabla}\times\vec{B}_2 = \frac{4\pi}{c}\vec{j}_2$. Then

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\left[\underbrace{B_1^2}_{\text{self energy of } \vec{j}_1} + \underbrace{B_2^2}_{\text{self energy of } \vec{j}_2} + \underbrace{2\vec{B}_1\cdot\vec{B}_2}_{\text{interaction energy of } \vec{j}_1 \text{ with } \vec{j}_2}\right]$$

$$\mathcal{E}_{int} = \frac{1}{4\pi}\int d^3r\ \vec{B}_1\cdot\vec{B}_2$$

$$= \frac{1}{c}\int d^3r\ \vec{j}_1\cdot\vec{A}_2 = \frac{1}{c}\int d^3r\ \vec{j}_2\cdot\vec{A}_1$$

where $\vec{B}_1 = \vec{\nabla}\times\vec{A}_1$, $\vec{B}_2 = \vec{\nabla}\times\vec{A}_2$, by similar manipulations as earlier

integrals are over all space

Apply to the interaction energy of a magnetic dipole in an external $\vec{B}$ field.

$$\mathcal{E}_{int} = \frac{1}{c}\int d^3r\ \underbrace{\vec{j}_1}_{\text{current distribution of dipole}}\cdot\underbrace{\vec{A}_2}_{\text{vector potential of external } \vec{B} \text{ field}}$$

Assuming $\vec{A}$ varies slowly on length scale of $\vec{j}$, then expand $A_i(\vec{r}) = A_i(\vec{r}_0) + (\vec{r}-\vec{r}_0)\cdot\vec{\nabla}A_i(\vec{r}_0)$

$$\mathcal{E}_{int} = \frac{1}{c}\int d^3r\, \vec{j}_1 \cdot \vec{A}(\vec{r}_0)$$

$$+ \frac{1}{c}\int d^3r \sum_i j_{1i}(r-r_0)_j\, \partial_j A_i(\vec{r}_0)$$

~~Shift origin so origin at $\vec{r}_0$ ~ $\vec{r}$ now measures distance~~

From ~~magnetostatic~~ computation of magnetic dipole moment we had $\int d^3r\, \vec{j} = 0$ for magnetostatics

$\Rightarrow$ 1st term above vanishes. So does the piece of 2nd term $\left(\int d^3r\, j_{1i}\right) r_{0j}\, \partial_j A_i(\vec{r}_0)$

We are left with

$$\mathcal{E}_{int} = \left[\frac{1}{c}\int d^3r\; j_{1i}\, r_j\right]\partial_j A_i(\vec{r}_0)$$

summation over repeated indices is implied

From computation of magnetic dipole approx we had

$$\int d^3r\, j_{1i} r_j = -\int d^3r\, j_{1j} r_i$$

$$= \frac{1}{2}\int d^3r\left[j_{1i}r_j - j_{1j}r_i\right]$$

Recall:
$$\boxed{\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r}\times\vec{j}}$$

$$= \frac{1}{2}\varepsilon_{kij}\int d^3r\,(\vec{j}\times\vec{r})_k$$

$$\Rightarrow \frac{1}{c}\int d^3r\, j_{1j} r_i = -\varepsilon_{kij}\, m_k$$ ← mag dipole moment

$$\mathcal{E}_{int} = -m_k \varepsilon_{kij} \partial_j A_i = m_k \varepsilon_{kji} \partial_j A_i$$

$$= \vec{m} \cdot (\vec{\nabla} \times \vec{A}) = \vec{m} \cdot \vec{B} = \mathcal{E}_{int}$$

This is opposite in sign to what we found earlier!

Why the difference?

① when we integrate the work done against the magnetostatic force to move $\vec{m}$ into position from infinity we found the energy

$$U = -\vec{m} \cdot \vec{B}$$

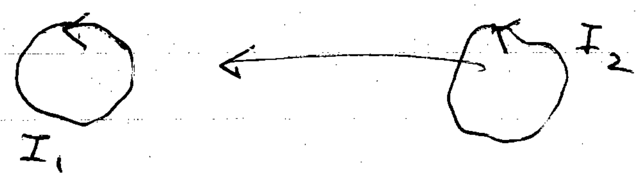
② when we compute the interaction energy from

$$\mathcal{E}_{int} = \frac{1}{c}\int d^3r\, \vec{j}_1 \cdot \vec{A}_2 = \frac{1}{c^2}\int d^3r \int d^3r' \frac{\vec{j}_1(\vec{r}) \cdot \vec{j}_2(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

we found the energy $\mathcal{E}_{int} = +\vec{m} \cdot \vec{B}$

To see which is correct, let us consider computing the interaction energy ② directly via method ①.

Consider two loops with currents I_1 and I_2
What is the work done to move loop 2 in from infinity to its final position with respect to loop 1?



Magnetostatic force on loop 2 due to loop 1 is

$$\vec{F} = \frac{I_2}{c} \oint_2 d\vec{l}_2 \times \vec{B}_1$$ Lorentz force
$\vec{B}_1$ is magnetic field from loop 1

$$\vec{B}_1(\vec{r}) = \frac{I_1}{c} \oint_1 d\vec{l}_1 \times \frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3}$$ Biot - Savart law

$$F = \frac{I_1 I_2}{c^2} \oint_2 \oint_1 d\vec{l}_2 \times \frac{(d\vec{l}_1 \times (\vec{r}_2-\vec{r}_1))}{|\vec{r}_2-\vec{r}_1|^3}$$

use triple product rule

$$d\vec{l}_2 \times [d\vec{l}_1 \times (\vec{r}_2-\vec{r}_1)] = d\vec{l}_1 [d\vec{l}_2 \cdot (\vec{r}_2-\vec{r}_1)] - (\vec{r}_2-\vec{r}_1)(d\vec{l}_1 \cdot d\vec{l}_2)$$

from the 1st term

$$\oint_2 d\vec{l}_2 \cdot \frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3} = -\oint_2 d\vec{l}_2 \cdot \vec{\nabla}_2 \left(\frac{1}{|\vec{r}_2-\vec{r}_1|}\right) = 0$$

as integral of gradient around closed loop always vanishes!

So

$$\vec{F} = -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

write $\vec{r}_2 = \vec{R} + \delta\vec{r}_2$ where $\vec{R}$ is center of loop 2

use $$\frac{\vec{R} + \delta\vec{r}_2 - \vec{r}_1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|^3} = -\vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$$

$$\vec{F} = \frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \, \vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$$

to move loop 2 we need to apply a mechanical force equal and <u>opposite</u> to the above magnetostatic force. Therefore the work we do in moving loop 2 from infinity to its final position at $\vec{R}_0$ is

$$W_{mech} = -\int_{\infty}^{\vec{R}_0} \vec{F} \cdot d\vec{R} = -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \int_{\infty}^{\vec{R}_0} d\vec{R} \cdot \vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$$

$$= -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|} \qquad \text{where } \vec{r}_2 = \vec{R}_0 + \delta\vec{r}_2$$

$$= -\frac{1}{c^2} \int d^3 r_1 \int d^3 r_2 \frac{\vec{j}_1(\vec{r}_1) \cdot \vec{j}_2(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|}$$

Note the minus sign!

this is just the <u>negative</u> of the interaction energy!!

$$= -M_{12} I_1 I_2$$

mutual inductance

Why the minus sign!

The minus sign we have here is the same minus sign we got when we found $U = -\vec{m}\cdot\vec{B}$ by integrating the force on the magnetic dipole.

Why dont we get $+\frac{1}{c^2}\int d^3r_1\, d^3r_2\, \frac{\vec{J}_1(r_1)\cdot\vec{J}_2(r_2)}{|\vec{r}_2-\vec{r}_1|}$

with the <u>plus</u> sign we expect from $\mathcal{E} = \frac{1}{8\pi}\int d^3r\, B^2$?

<u>Answer:</u> we have left something out!

<u>Faraday's law</u> - when we move loop 2, the magnetic flux through loop 2 changes. This $\frac{d\Phi}{dt}$ creates an emf $= \oint d\vec{l}\cdot\vec{E}$ around the loop that would tend to change the current in the loop. If we are to keep the current fixed at constant I_2 then there must be a battery in the loop that does work to counter this induced emf ("electromotive force"). Similarly, the flux through loop 1 is changing and a battery does work to keep I_1 constant. We need to add this work done by the batteries to the mechanical work computed above.

emf induced in loop 1 $\quad \mathcal{E}_1 = \oint_1 d\vec{l}_1\cdot\vec{E}_2$

emf induced in loop 2 $\quad \mathcal{E}_2 = \oint_2 d\vec{l}_1\cdot\vec{E}_1$

$\Big\}$ integrations in direction of current

Faraday $\quad \mathcal{E}_1 = \frac{-d\Phi_1}{c\,dt} \qquad \Phi_1$ = flux through loop 1

$\mathcal{E}_2 = \frac{-d\Phi_2}{c\,dt} \qquad \Phi_2$ = flux through loop 2

To keep the current constant, the batteries need to provide an emf that counters these Faraday induced emf's. The work done by the batteries per unit time is therefore

$$\frac{dW_{battery}}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2$$

(check units: $\mathcal{E} I$ is [length]$\cdot$[E]$\cdot$[$\mathcal{Q}$/s]
= [length]$\cdot$[force/s]
= energy/s)

$$\frac{dW_{battery}}{dt} = \frac{d\Phi_1}{c\,dt} I_1 + \frac{d\Phi_2}{c\,dt} I_2$$

$$W_{battery} = \int_0^T dt \left(\frac{d\Phi_1}{c\,dt} I_1 + \frac{d\Phi_2}{c\,dt} I_2 \right)$$

where $t=0$ loop 2 is at infinity
$t=T$ loop 2 is at final position
I_1, I_2 kept constant as loop moves

$$W_{battery} = \frac{1}{c}\Phi_1 I_1 + \frac{1}{c}\Phi_2 I_2$$

where Φ_1 and Φ_2 are fluxes in final position, and we assumed that fluxes $=0$ at infinity

$$\Phi_1 = c M_{12} I_2$$
$$\Phi_2 = c M_{21} I = c M_{12} I_1 \quad \text{as } M_{12} = M_{21}$$

$$\Rightarrow W_{battery} = 2 M_{12} I_1 I_2$$

add this to the mechanical work

$$W_{total} = W_{mech} + W_{battery} = -M_{12} I_1 I_2 + 2 M_{12} I_1 I_2$$

$$= M_{12} I_1 I_2 = +\frac{1}{c^2}\int d^3r_1\, d^3r_2\, \frac{\vec{J}_1(\vec{r}_1)\cdot\vec{J}_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$$

we get back the correct interaction energy!

<u>Conclusion</u> : The magnetostatic interaction energy $\frac{1}{c^2}\int d^3r_1\, d^3r_2\, \frac{\vec{J}_1(\vec{r}_1)\cdot\vec{J}_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$ includes the work done to maintain the currents stationary as the current distributions move.

When we computed the interaction energy of a <u>current loop</u> dipole $\vec{m}$ and find

$$\mathcal{E}_{int} = +\vec{m}\cdot\vec{B}$$

this includes the energy needed to maintain the constant current producing the constant $\vec{m}$

When we integrated the force on the dipole to find the potential energy

$$U = -\vec{m}\cdot\vec{B}$$

this <u>did not</u> include the energy needed to maintain ~~the~~ constant current that creates $\vec{m}$.
This is the correct energy expression to use when $\vec{m}$ comes from <u>intrinsic</u> <u>magnetic</u> <u>moments</u> due to particles intrinsic spin, which cannot be viewed as arising from a current loop!

Electromagnetic Waves in a vacuum

No sources $\vec{j}=0$, $\rho=0$

1) $\vec{\nabla}\cdot\vec{E}=0$ 3) $\vec{\nabla}\cdot\vec{B}=0$

2) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{c\,\partial t}$ 4) $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$$\nabla\times(2) \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{0 \text{ by } (1)}) - \nabla^2\vec{E} = -\frac{\partial}{c\,\partial t}(\vec{\nabla}\times\vec{B})$$

$$-\nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{c\,\partial t}\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Similarly

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

} wave equation
wave speed is c.

[Note: in MKS units, above wave equation looks like

$$\nabla^2\vec{E} - \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

It was noticed that the speed of electromagnetic wave,

$\frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8$ m/s was the same as the speed of light! This observation was a key element in showing that light was in fact electromagnetic wave]

Harmonic

Plane waves

$$\left.\begin{aligned}\vec{E}(\vec{r},t) &= \mathrm{Re}\left[\vec{E}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]\\ \vec{B}(\vec{r},t) &= \mathrm{Re}\left[\vec{B}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]\end{aligned}\right\}\ \text{complex exponential form}$$

$\vec{k}$ is wave vector

ω is angular frequency

$\nu = \omega/2\pi$ is frequency

$T = 1/\nu$ is period

$\lambda = \frac{2\pi}{|k|}$ is wavelength

$\left.\begin{matrix}|\vec{E}_k|\\ |\vec{B}_k|\end{matrix}\right\}$ is amplitude

$\vec{E}(\vec{r}+\lambda\hat{k},t) = \vec{E}(\vec{r},t)$ periodic in space with period λ

$\vec{E}(\vec{r},t+T) = \vec{E}(r,t)$ periodic in time with period T

"plane wave" $\Rightarrow$ $\vec{E}(\vec{r},t)$ is constant in space on planes with normal $\hat{n} \parallel \vec{k}$.

properties of EM plane waves

$$\vec{\nabla}\cdot\vec{E} = 0 \quad\Longrightarrow\quad \mathrm{Re}\left[\vec{E}_k\cdot\vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$= \mathrm{Re}\left[i\vec{E}_k\cdot\vec{k}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \vec{E}_k\cdot\vec{k} = 0$$

amplitude is orthogonal to $\vec{k}$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad\rightarrow\quad \vec{B}_k\cdot\vec{k} = 0$$

amplitude orthogonal to $\vec{k}$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \quad Re\left[\vec{\nabla} \times \vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right] = Re\left[\frac{1}{c} \vec{E}_k \frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right]$$

$$\Rightarrow Re\left[-\vec{B}_k \times \vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right] = Re\left[-\frac{i\omega}{c} \vec{E}_k \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right]$$

$$\Rightarrow \quad Re\left[i\vec{k} \times \vec{B}_k \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right] = Re\left[-\frac{i\omega}{c} \vec{E}_k \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right]$$

$$\Rightarrow \quad \vec{k} \times \vec{B}_k = -\frac{\omega}{c} \vec{E}_k$$

$$\vec{k} \times \vec{k} \times \vec{B}_k = -k^2 \vec{B}_k = -\frac{\omega}{c} \vec{k} \times \vec{E}_k$$

$$\underline{\vec{B}_k = \frac{\omega}{ck^2} \vec{k} \times \vec{E}_k}$$

Finally

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\Rightarrow \quad Re\left[\vec{E}_k \nabla^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \frac{\vec{E}_k}{c^2} \frac{\partial^2}{\partial t^2} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right] = 0$$

$$\Rightarrow \quad Re\left[\vec{E}_k (-k^2) e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\omega^2}{c^2} \vec{E}_k \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right] = 0$$

$$\Rightarrow \quad k^2 = \frac{\omega^2}{c^2}$$

$$\boxed{\omega = \pm kc}$$ <u>dispersion relation</u>

insert in above

$$\vec{B}_k = \hat{k} \times \vec{E}_k \qquad \hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

$$\Rightarrow \quad |\vec{B}_k| = |\vec{E}_k|$$