assume μ is true constant - not freq dependent
dielectric response is $\vec{D}_\omega = \varepsilon(\omega)\vec{E}_\omega$

Then for the Fourier amplitudes of the fields, Maxwell's Equations become

transverse polarized

1) $i\vec{k}\cdot\vec{D}_\omega = i\,\varepsilon(\omega)\,\vec{k}\cdot\vec{E}_\omega = 0 \quad\Rightarrow \boxed{\vec{k}\perp\vec{E}_\omega}$ (unless $\varepsilon(\omega)=$

2) $i\vec{k}\cdot\vec{B}_\omega = 0 \quad\Rightarrow \boxed{\vec{k}\perp\vec{B}_\omega}$

3) $i\vec{k}\times\vec{E}_\omega = i\frac{\omega}{c}\vec{B}_\omega$

4) $i\vec{k}\times\vec{H}_\omega = -i\frac{\omega}{c}\vec{D}_\omega \Rightarrow \frac{i\vec{k}}{\mu}\times\vec{B}_\omega = -i\frac{\omega}{c}\varepsilon(\omega)\vec{E}_\omega$

$$\vec{k}\times(3) = i\vec{k}\times(\vec{k}\times\vec{E}_\omega) = i\frac{\omega}{c}\vec{k}\times\vec{B}_\omega$$

$$\Rightarrow -ik^2\vec{E}_\omega = -i\frac{\omega^2}{c^2}\varepsilon(\omega)\mu\vec{E}_\omega \qquad \text{using (4)}$$

$$\boxed{k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)\mu}$$ <u>dispersion relation</u>

<u>Note</u>: $\frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon(\omega)\mu}}$ varies with ω.
There is not a single phase velocity.
$\Rightarrow \vec{E}$ is <u>not</u> in general a solution of a wave equation - different frequencies travel with different spee

Since $\varepsilon(\omega)$ is complex $\quad \varepsilon(\omega) = \varepsilon_1(\omega) + i\varepsilon_2(\omega)$

$\Rightarrow$ wave vector also complex $\quad$ For $\vec{k} = k\hat{z}$

$$k = k_1 + ik_2 = \pm \frac{\omega}{c}\sqrt{\mu}\,\sqrt{\varepsilon_1 + i\varepsilon_2}$$

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_\omega e^{i[(k_1+ik_2)z - \omega t]}$$
$$= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

k_1 determines the oscillation of the wave

k_2 determines the decay or <u>attenuation</u> of the wave as it propagates into the material

<u>phase velocity</u> $\quad v_p \equiv \dfrac{\omega}{k_1}$

<u>index of refraction</u> $\quad n = \dfrac{c}{v_p} = \dfrac{ck_1}{\omega}$

<u>group velocity</u> $v_g \quad \dfrac{1}{v_g} = \dfrac{dk_1}{d\omega}$

<u>Magnetic field</u>: $\quad \vec{B}_\omega = \dfrac{\vec{k}}{\omega} \times \vec{E}_\omega$

for $\vec{k} = k\hat{z}$, $\quad \vec{B}_\omega = \dfrac{(k_1 + ik_2)}{\omega}\hat{z} \times \vec{E}_\omega$

if $k_1 + ik_2 = \sqrt{k_1^2 + k_2^2}\, e^{i\delta} \qquad \delta = \arctan($
$= |k| e^{i\delta}$

$$\vec{B}_\omega = \frac{|k|}{\omega}\hat{z} \times \vec{E}_\omega e^{i\delta}$$

phase shift

$$\vec{B}(\vec{r},t) = \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{-k_2 z}\, e^{i(k_1 z - \omega t + \delta)}$$

<u>Physical fields</u> - take real parts

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2 z}\cos(k_1 z - \omega t)$$

$$\vec{B}(\vec{r},t) = (\hat{z}\times\vec{E}_\omega)\frac{|k|}{\omega}\, e^{-k_2 z}\cos(k_1 z - \omega t + \delta)$$

<u>Conclusions</u>

1) $\vec{E}$ and $\vec{B}$ $\perp$ $\vec{k}$ transverse polarized
2) $\vec{E} \perp \vec{B}$
3) amplitude ratio $\frac{|\vec{B}|}{|\vec{E}|} = \frac{|k|}{\omega} = \sqrt{|\varepsilon(\omega)|\mu}$
4) $\vec{B}$ is shifted in phase with respect to $\vec{E}$ by phase shift $\delta = \arctan(k_2/k_1)$
5) waves decay as they propagate $e^{-k_2 z}$

(4–5: consequence of complex $\varepsilon(\omega)$)

If $\varepsilon_2 = 0$, ie $\varepsilon(\omega)$ is real, <u>and</u> if $\varepsilon > 0$, then $k_2 = 0 \Rightarrow$ no decay, no phase shift

consequences of frequency dependence of $\varepsilon(\omega)$:

6) $\vec{E}(t)$ and $\vec{D}(t)$ non locally related in time
7) waves of different ω travel with different $v_p = \omega/k$
8) <u>dispersion</u> - wave pulses do <u>not</u> travel with v_p and they spread as the propagate
pulses travel with group velocity $v_g = \frac{d\omega}{dk}$
(see Quantum Mechanics discussion)
$v_g < v_p$ "normal dispersion"
$v_g > v_p$ "anomalous dispersion"

index of refraction

$$\frac{1}{v_g} = \frac{dk_1}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c} n\right]$$

$$\frac{1}{v_g} = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c}\,\omega\frac{dn}{d\omega}}$$

$\Rightarrow$ when $\begin{cases} \frac{dn}{d\omega} > 0, & v_g < v_p \quad \text{normal dispersion} \\ \frac{dn}{d\omega} < 0, & v_g > v_p \quad \text{anomalous dispersion} \end{cases}$

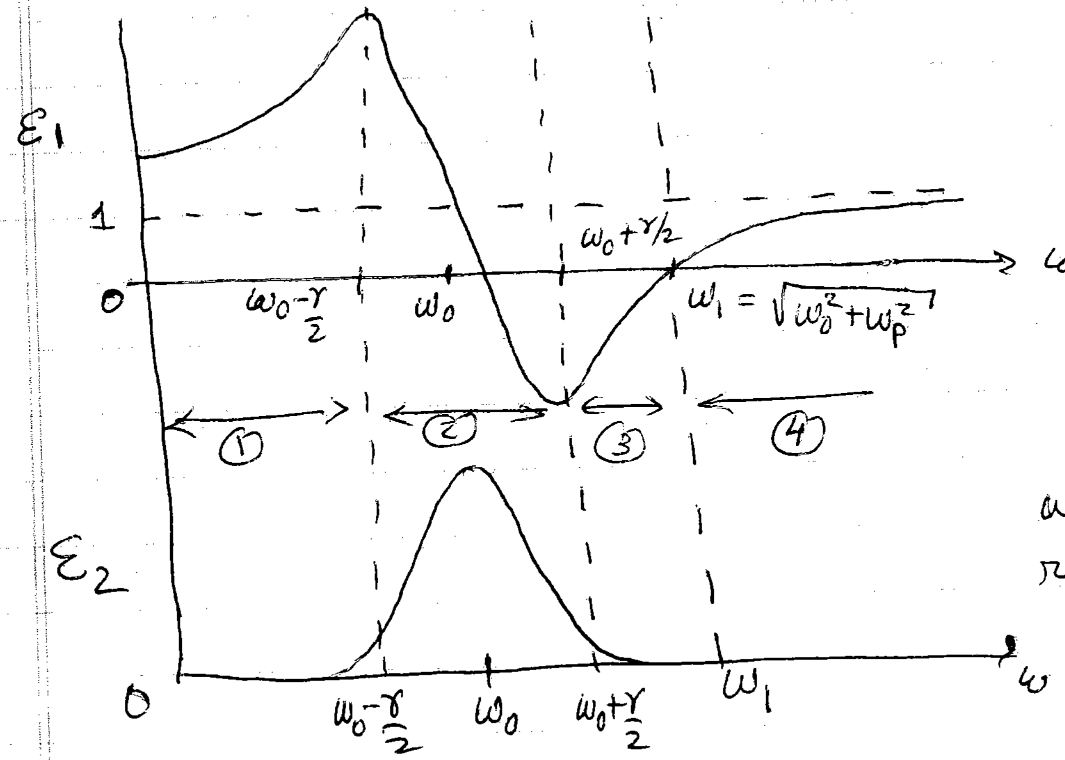
For our simple model: $\varepsilon = 1 + 4\pi\chi \simeq 1 + 4\pi M\alpha$

$$\varepsilon(\omega) = 1 + \frac{4\pi M e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$\varepsilon_1 = 1 + \frac{4\pi M e^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\varepsilon_2 = \frac{4\pi M e^2}{m} \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

Define $\omega_p = \sqrt{\frac{4\pi M e^2}{m}}$ the "plasma frequency"



width $\sim \gamma$

resonance at $\bar{\omega} \approx \omega_0$

$$k = k_1 + i k_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\varepsilon_1 + i\varepsilon_2}$$

$$k^2 = k_1^2 - k_2^2 + 2 i k_1 k_2 = \frac{\omega^2}{c^2} \mu (\varepsilon_1 + i\varepsilon_2)$$

equate real and imaginary pieces and solve for k_1 and k_2

$$k_1 = \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

Regions of different behavior

Regions 1 and 4 — transparent propagation

$\varepsilon_1 > 0 \quad , \quad \varepsilon_1 \gg \varepsilon_2$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_1\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\varepsilon_1 + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1}\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu\varepsilon_1} \quad + \text{ small correction}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_1\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1}\right]^{1/2} = k_1\left(\frac{\varepsilon_2}{2\varepsilon_1}\right) \ll$$

So $k_2 \ll k_1$ small attenuation

$\Rightarrow$ medium is transparent

Note: $v_p = \frac{\omega}{k_1} = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon_1\mu}}$

in region 1, $\varepsilon_1 > 1 \Rightarrow v_p < c$

in region 4, $\varepsilon_1 < 1 \Rightarrow v_p > c$!

but $v_g < c$ always !

Region ② $\quad \omega \approx \omega_0 \quad$ resonant absorption

$$\varepsilon_2 \approx \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{\omega_0}{\gamma}\right) \gg 1$$ for a sharp resonance with $\gamma \ll \omega_0$

$$\varepsilon_1 \approx 1$$

so $\quad \varepsilon_2 \gg \varepsilon_1$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2\left(1 + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right) + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2 + \frac{1}{4}\frac{\varepsilon_1^2}{\varepsilon_2} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\tfrac{1}{2}\varepsilon_2}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2\left(1 + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right) - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2 + \frac{1}{4}\frac{\varepsilon_1^2}{\varepsilon_2} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\tfrac{1}{2}\varepsilon_2}$$

$k_1 \approx k_2 \quad$ strong attenuation

wave excits atoms at resonance $\Rightarrow$ large atomic displacements $\Rightarrow$ media absorbs most energy from the wave $\Rightarrow$ wave decays rapidly, decreases factor $\frac{1}{e^{2\pi}}$ within one wavelength of propagation.

Region ③ $\varepsilon_1 < 0, \quad |\varepsilon_1| \gg \varepsilon_2$

total reflection

width of region ③ is

$$\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{M}$$

increases with atomic density as $\omega_p \gg \omega_0$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\varepsilon_1| + \frac{1}{4}\frac{\varepsilon_2^2}{|\varepsilon_1|} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

cancel as $|\varepsilon_1| = -\varepsilon_1$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu|\varepsilon_1|}\ \frac{\varepsilon_2}{2|\varepsilon_1|}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\varepsilon_1| + \frac{1}{4}\frac{\varepsilon_2^2}{|\varepsilon_1|} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu|\varepsilon_1|}$$

$$\frac{k_2}{k_1} = \frac{2|\varepsilon_1|}{\varepsilon_2} \gg 1$$

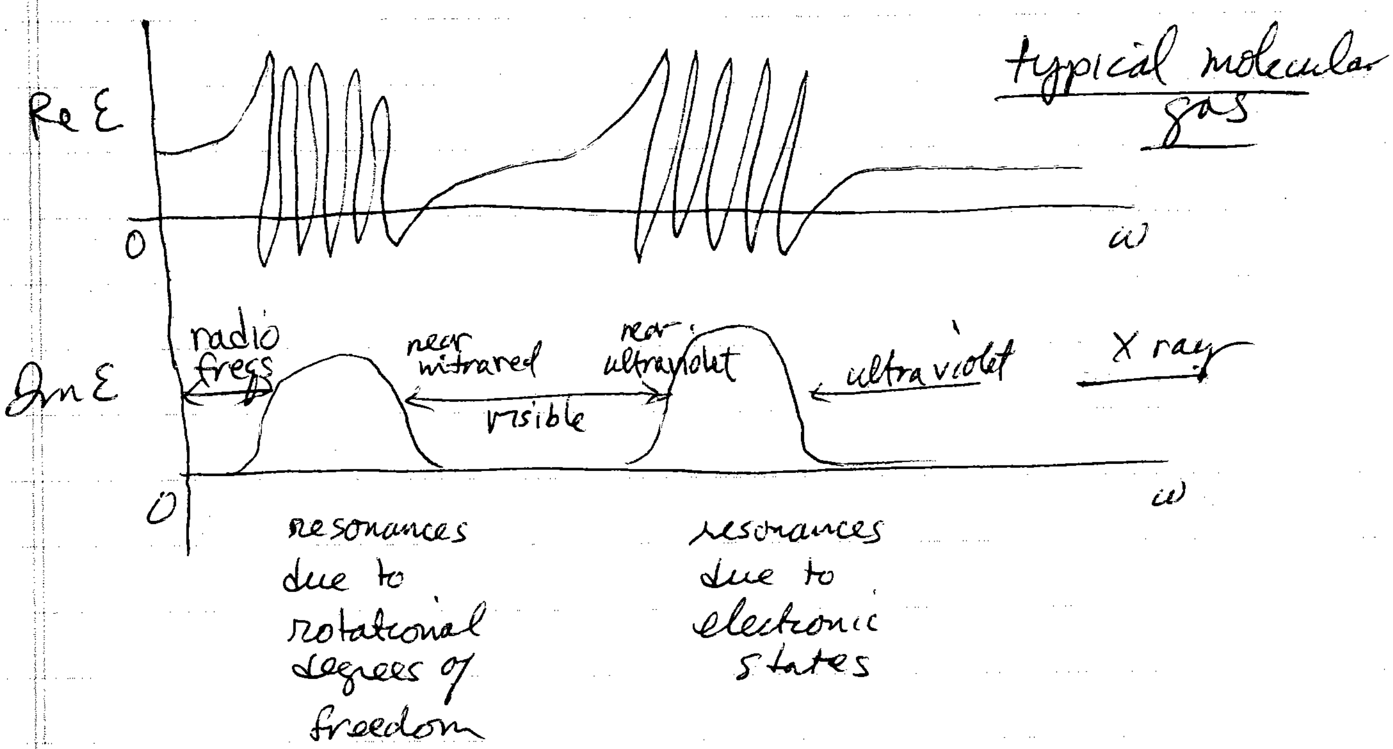
wave vector is almost pure imaginary
wave decays exponentially to zero in much less than one wavelength.

we will see this corresponds to total reflection
Since $\omega \gg \omega_0$, we are not at resonance so material is not absorbing much energy from wave. The strong attenuation is due to the destructive interference between the wave and the induced fields of the polarized atoms

Our single model had a single resonance at ω_0. A more realistic model for molecules has many bands of resonances due to rotational, vibrational, and electronic modes of excitation

$$\varepsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where $\hbar\omega_i$ are spacings between energy levels with allowed electric dipole transitions



$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

$$= 4.4\times10^{16}\sqrt{\frac{n}{n_A}}\ \sec^{-1}, \quad n_A = 6\times10^{23}/\mathrm{cm}^3$$

$$\Rightarrow \hbar\omega_p = 185\sqrt{\frac{n}{n_A}}\ \mathrm{eV}$$

For H_2O $\frac{n}{n_A} \sim 0.05$

$$\hbar\omega_p \sim 40\ \mathrm{eV}$$

For typical metal $\frac{n}{n_A} \sim 0.1$

$$\hbar\omega_p \sim 58\ \mathrm{eV}$$

compared to $\hbar\omega_0 \sim \mathrm{eV}$

EM waves in Conductors

Conduction electrons are mobil, not bound
$\Rightarrow$ we have to include the $\vec{J}_f$ and ρ_f from them.

Simple classical model for electron motion – "Drude" Model

$$m\ddot{\vec{r}} = -e\vec{E}(t) - \frac{m}{\tau}\dot{\vec{r}}$$

external E field

damping force due to collisions
τ is "relaxation time"

$$\vec{E} = \vec{E}_\omega e^{-i\omega t} \Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t} \text{ solution}$$

plug in to get

$$\left(-\omega^2 - \frac{i\omega}{\tau}\right)\vec{r}_\omega = -\frac{e}{m}\vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m}\frac{1}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega$$

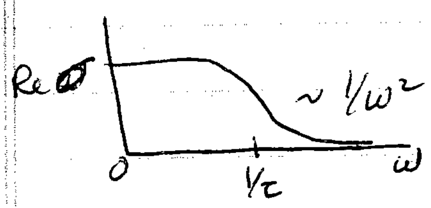
current is $\vec{J}_f = -en\dot{\vec{r}}_\omega = -en(-i\omega)\vec{r}_\omega$

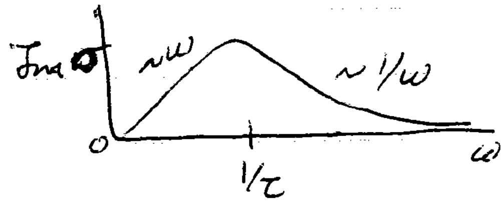
density of electrons

$$\vec{J}_f = \frac{ne^2}{m}\frac{i\omega}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}\vec{E}$$

$$\vec{J}_f = \sigma(\omega)\,\vec{E}_\omega$$

conductivity $\boxed{\sigma(\omega) = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}}$





$$\mathrm{Re}\,\sigma = \frac{\sigma_0}{1+\omega^2\tau^2} \qquad \mathrm{Im}\,\sigma = \sigma_0 \frac{\omega\tau}{1+\omega^2\tau^2}$$

$$\sigma_0 = \sigma(0) = \frac{ne^2\tau}{m}$$ dc conductivity

Charge density ρ_f given by charge conservation law.
For plane waves

$$\rho_f = \rho_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad , \quad \vec{J}_f = \vec{J}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{J}_f \Rightarrow -i\omega\rho_\omega = -i\vec{k}\cdot\vec{J}_\omega$$

$$\rho_\omega = \frac{\vec{k}\cdot\vec{J}_\omega}{\omega} = \sigma(\omega)\frac{\vec{k}\cdot\vec{E}_\omega}{\omega}$$

Maxwell Equations

1) $\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$ 2) $\vec{\nabla}\cdot\vec{B} = 0$

3) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

Assume $\vec{H} = \vec{B}/\mu$, μ constant

$\vec{D}_\omega = \varepsilon_b(\omega)\vec{E}_\omega$ $\varepsilon_b(\omega)$ is dielectric function from the bound charges

$\vec{J}_\omega = \sigma(\omega)\vec{E}_\omega$ $\sigma(\omega)$ is conductivity from free charges

$\rho_\omega = \frac{\sigma}{\omega}\vec{k}\cdot\vec{E}_\omega$