For harmonic plane wave solutions $\vec{E} = E_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ etc.

1) $\Rightarrow \; i\vec{k}\cdot\vec{D}_\omega = i\vec{k}\cdot\varepsilon_b E_\omega = 4\pi\rho_\omega = 4\pi\sigma\frac{\vec{k}\cdot\vec{E}_\omega}{\omega}$

$\Rightarrow \; i\vec{k}\cdot\vec{E}_\omega\left(\varepsilon_b + \frac{4\pi i\sigma}{\omega}\right) = 0$

2) $\Rightarrow \; i\mu\vec{k}\cdot\vec{H}_\omega = 0$

3) $\Rightarrow \; i\vec{k}\times\vec{E}_\omega = \frac{i\omega\vec{B}_\omega}{c} = \frac{i\omega\mu H_\omega}{c}$

4) $\Rightarrow \; i\vec{k}\times\vec{H}_\omega = \frac{4\pi}{c}\vec{J}_\omega - \frac{i\omega}{c}\vec{D}_\omega$

$= \frac{4\pi}{c}\sigma\vec{E}_\omega - \frac{i\omega}{c}\varepsilon_b\vec{E}_\omega$

$= -\frac{i\omega}{c}\left(\varepsilon_b + \frac{4\pi i\sigma}{\omega}\right)\vec{E}_\omega$

Notice: all the equations above look exactly like what we had for the dielectric, provided we define

$$\boxed{\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{4\pi i\sigma(\omega)}{\omega}}$$

So all results for the dielectric case carry over to conductors, provided we make the above substitution. In particular

dispersion relation for transverse modes $\boxed{k^2 = \frac{\omega^2}{c^2}\mu\,\varepsilon(\omega)}$

The main difference between dielectrics & conductors has to do with the contribution that the $4\pi i\sigma/\omega$ makes to the real and imaginary parts of $\varepsilon(\omega)$.

For simple Drude model $\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau}$ $\sigma_0 = \frac{ne^2\tau}{m}$

① Low frequencies $\omega \ll 1/\tau$

$\varepsilon_b(\omega) \approx \varepsilon_b(0)$ real

$\sigma(\omega) \approx \sigma_0$ real

$\Rightarrow \boxed{\varepsilon(\omega) \simeq \varepsilon_b(0) + \frac{4\pi i\sigma_0}{\omega}}$ ← gives large ε_2 as $\omega \to 0$

⇒ strong dissipation

$\mathrm{Re}\,\varepsilon = \varepsilon_1$

$\mathrm{Im}\,\varepsilon = \varepsilon_2$

when $\frac{\varepsilon_2}{\varepsilon_1} = \frac{4\pi\sigma_0}{\omega\varepsilon_b(0)} \gg 1$ we call this regime a "good" conductor. conduction electrons dominate the response – waves strongly attenuated

when $\frac{\varepsilon_2}{\varepsilon_1} = \frac{4\pi\sigma_0}{\omega\,\varepsilon_b(0)} \ll 1$ we call this regime a "poor" conductor. little absorption of energy by conduction electrons. waves propagate

one always enters the "good" conductor region when ω gets sufficiently small.

wave vector:

$$k = \frac{\omega}{c}\sqrt{\mu\varepsilon}$$

for a good conductor where $\varepsilon_2 \gg \varepsilon_1$,

$$\varepsilon \sim i\varepsilon_2 = \frac{4\pi i \sigma_0}{\omega}$$

$$k = k_1 + i k_2 = \frac{\omega}{c}\sqrt{\mu \frac{4\pi i \sigma_0}{\omega}} \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$k_1 = k_2 = \frac{\omega}{c}\sqrt{\frac{4\pi\mu\sigma_0}{2\omega}} = \frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

for $\vec{k} = k\hat{z}$,

$$\vec{E} = \vec{E}_\omega e^{i(kz-\omega t)} = \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

$$\delta \equiv 1/k_2 = \frac{c}{\sqrt{2\pi\mu\sigma_0\omega}}$$

"skin depth" distance wave propagates into conductor

$\delta \sim 1/\sqrt{\omega}$ increases as ω decreases

ϕ phase shift between oscillations of $\vec{E}$ and $\vec{H}$

$$\phi = \arctan(k_2/k_1) \simeq \arctan(1) = 45^\circ$$

Amplitude ratio

$$\frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{c|k|}{\omega\mu} = \frac{\sqrt{2}\,c}{\omega\mu}k_1$$

$$= \frac{\sqrt{2}\,c}{\omega\mu}\frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

$$= \sqrt{\frac{4\pi\sigma_0}{\omega\mu}} \sim 1/\sqrt{\omega}$$

as $\omega \to 0$, most of the energy of the wave is carried by the magnetic field part

(2) high frequencies $\omega \gg 1/\tau$, $\omega \gg \omega_0$

$$\varepsilon_b(\omega) \approx 1$$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{i n e^2 \tau}{m\omega\tau} = \frac{i n e^2}{m\omega}$$ pure imag, indep of τ

$$\varepsilon(\omega) \simeq 1 + \frac{4\pi i \sigma}{\omega} \simeq 1 - \frac{4\pi n e^2}{m\omega^2}$$

$$\boxed{\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}}$$

$\omega_p \equiv \sqrt{\frac{4\pi n e^2}{m}}$ plasma freq of the conduction electrons

$\varepsilon(\omega)$ is real

1) If $\omega > \omega_p$ then $\varepsilon > 0$

$\Rightarrow$ transparent propagation

$k = k_1 = \frac{\omega}{c}\sqrt{\mu\varepsilon}$ is pure real

$k_2 \approx 0$

2) If $\omega < \omega_p$ then $\varepsilon < 0$

$\Rightarrow$ total reflection

$k_1 \approx 0$ k is pure imaginary

$k = k_2 = \frac{\omega}{c}\sqrt{\mu|\varepsilon|}$

ω_p gives cross over between total reflection and transparent propagation for typical metals

$\tau \sim 10^{-14}$ sec

$\omega_p \sim 10^{16}$ sec^{-1}

$\lambda_p = \frac{2\pi c}{\omega_p} \sim 3 \times 10^3$ Å (visible is $\lambda \sim 5 \times 10^3$ Å)

Example: The ionesphere is a layer of charged gas surrounding the earth. In many respects the charged particles of the ionesphere behave like conduction electrons in a metal. The plasma freq of the ionesphere is such that

for AM radio $\omega_{AM} < \omega_p \Rightarrow$ AM radio signals reflected back to earth

for FM radio $\omega_{FM} > \omega_p \Rightarrow$ FM radio signals propagate through ionesphere into space

Explains why you can pick up AM stations from far away - they get reflected back
But you can only pick up local FM stations.

Longitudinal modes in conductors

ie $\vec{H}_\omega$ or $\vec{E}_\omega$ not $\perp \vec{k}$

magnetic field

$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow i\mu\vec{k}\cdot\vec{H}_\omega=0 \Rightarrow \vec{H}_\omega\perp\vec{k}$ transverse

or $\vec{k}=0$ spatially uniform $\vec{H}$

if $\vec{k}=0$ then Faraday

$$\underbrace{i\vec{k}\times\vec{E}_\omega}_{=0 \text{ as } \vec{k}=0} = \frac{i\omega\mu}{c}\vec{H}_\omega = 0 \Rightarrow \omega=0$$

So only possible longitudinal $\vec{H}$ is spatially uniform, constant in time.

electric field

$\vec{\nabla}\cdot\vec{D}=4\pi\rho_f \Rightarrow i\varepsilon(\omega)\vec{k}\cdot\vec{E}_\omega=0 \Rightarrow \vec{E}_\omega\perp\vec{k}$ transverse

or $\varepsilon(\omega)=0$

If $\vec{E}_\omega \parallel \vec{k}$ but $\varepsilon(\omega)=0$, then can satisfy all other maxwell equations.

$$i\vec{k}\times\vec{E}_\omega = \frac{i\omega\mu}{c}\vec{H}_\omega \Rightarrow \vec{H}_\omega=0$$

$$\Rightarrow \underbrace{i\mu\vec{k}\cdot\vec{H}_\omega}_{=0 \text{ as } \vec{H}_\omega=0}=0 \quad \text{and} \quad \underbrace{i\vec{k}\times\vec{H}_\omega}_{=0} = \underbrace{-\frac{i\omega}{c}\varepsilon(\omega)\vec{E}_\omega}_{=0 \text{ as } \varepsilon(\omega)=0}$$

So we **can** have longitudinal electric field oscillation when $\varepsilon(\omega)=0$

low freq $\quad \omega \ll \omega_0 \quad \omega\tau \ll 1$

$$\varepsilon \simeq \varepsilon_b(0) + \frac{4\pi i \sigma_0}{\omega}$$

$$\varepsilon(\omega) = 0 \quad \text{when} \quad \omega = -\frac{4\pi i \sigma_0}{\varepsilon_b(0)}$$

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_\omega e^{-\frac{4\pi\sigma_0}{\varepsilon_b(0)}t} e^{i\vec{k}\cdot\vec{r}}$$

If set up a longitudinal $\vec{E}$ field, it decays to zero exponentially with ~~time const~~ decay time $\varepsilon_b(0)/4\pi\sigma_0$. This is consistant with assumption the $\vec{E}=0$ inside a conductor for electrostatics

in statics $\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{E} \sim -i\vec{k}\phi_k e^{i\vec{k}\cdot\vec{r}}$ is longitud.

high freq $\quad \omega \gg 1/\tau, \quad \omega \gg \omega_0$

$$\varepsilon(\omega) \approx 1 + \frac{4\pi i\sigma}{\omega} = 1 - \frac{\omega_p^2}{\omega^2} \qquad \omega_p^2 = \frac{4\pi n e^2}{m}$$

$\varepsilon = 0$ when $\omega = \omega_p$

So we have oscillatory longitudinal $\vec{E}$ only when $\omega = \omega_p$, independent of $\vec{k}$.

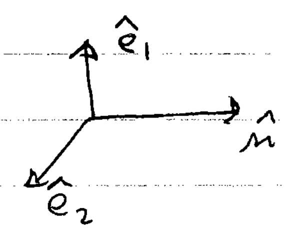
$$\vec{E} = \vec{E}_\omega e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_p t}$$

This is called a plasma oscillation. When one quantizes this oscillatory mode, it is called a plasmon

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \rho = \frac{i\vec{k}\cdot\vec{E}_\omega}{4\pi} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_p t}$ $\leftarrow$ plasma osc. is a charge density oscillation

Polarization

Consider transverse wave propagating in direction $\hat{n}$
ie $\vec{k} = k\hat{n}$.



$$\hat{e}_1 \times \hat{e}_2 = \hat{n}$$
$$\hat{n} \times \hat{e}_1 = \hat{e}_2$$
$$\hat{e}_2 \times \hat{n} = \hat{e}_1$$

A general solution for a transverse wave has the form

$$\vec{E}(\vec{r},t) = Re\left\{ (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

$$\vec{H}(\vec{r},t) = \frac{c}{\omega\mu} Re\left\{ k\,\hat{n} \times (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

$$= \frac{c}{\mu\omega} Re\left\{ k\,(E_1 \hat{e}_2 - E_2 \hat{e}_1)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

So far we considered implicitly only the case
E_1, E_2 real constants $\Rightarrow |\vec{E}_\omega| = \sqrt{E_1^2 + E_2^2}$
But most general case is

$$E_1 = E\cos\theta\, e^{i\chi_1}$$
$$E_2 = E\sin\theta\, e^{i\chi_2}$$

} need not be real
can be complex with
relative phase difference

$$E^2 = |E_1|^2 + |E_2|^2$$

(for $\chi_1 = \chi_2 = 0$, θ is angle $\vec{E}_\omega$ makes with respect to $\hat{e}_1$.

define $\Phi = k_1 \hat{n}\cdot\vec{r} - \omega t$
$\tan\varphi = k_2 / k_1$

$$\vec{E} = E\, e^{-k_2 \hat{n}\cdot\vec{r}} \left[\hat{e}_1 \cos\theta \cos(\Phi + \chi_1) + \hat{e}_2 \sin\theta \cos(\Phi + \chi_2) \right]$$

$$\vec{H} = \frac{c|k|}{\omega\mu} E\, e^{-k_2 \hat{n}\cdot\vec{r}} \left[\hat{e}_2 \cos\theta \cos(\Phi + \chi_1 + \varphi) - \hat{e}_1 \sin\theta \cos(\Phi + \chi_2 + \varphi) \right]$$

<u>special cases</u>

1) $\underline{\chi_1 = \chi_2}$

$$\vec{E} = (\hat{e}_1 \cos\theta + \hat{e}_2 \sin\theta) E\, e^{-k_2 \hat{n}\cdot\vec{r}} \cos(\Phi + \chi_1)$$

$$\vec{H} = (-\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta) \frac{|k| E}{\omega\mu} e^{-k_2 \hat{n}\cdot\vec{r}} \cos(\Phi + \chi_1 + \varphi)$$

<u>Linearly polarized</u> – $\vec{E}$ and $\vec{H}$ point in fixed directions and are orthogonal. phase shift is φ

2) $\underline{\cos\theta = \frac{1}{\sqrt{2}},\ \sin\theta = \pm 1/\sqrt{2}}$, $\chi_1 = 0$, $\chi_2 = \chi$ (can always choose $\chi_1 = 0$ by shifting the time scale)

$$\vec{E}^{\pm} = \frac{E}{\sqrt{2}} \left(\hat{e}_1 \cos\Phi \pm \hat{e}_2 \cos(\Phi + \chi) \right)$$

Find locus of points that $\vec{E}$ sweeps out as Φ varies.

$$\vec{E}^{\pm} \cdot \hat{e}_1 = E_1^{\pm} = \frac{E}{\sqrt{2}} \cos\Phi$$
$$\vec{E}^{\pm} \cdot \hat{e}_2 = E_2^{\pm} = \pm \frac{E}{\sqrt{2}} \cos(\Phi + \chi)$$

Define $\Theta = \Phi + \chi/2$ so

$$E_1^{\pm} = \frac{E}{\sqrt{2}} \cos(\Theta - \chi/2)$$
$$E_2^{\pm} = \pm \frac{E}{\sqrt{2}} \cos(\Theta + \chi/2)$$

$$E_1^{\pm} = \frac{E}{\sqrt{2}} \cos\Theta \cos\chi/2 + \frac{E}{\sqrt{2}} \sin\Theta \sin\chi/2$$
$$E_2^{\pm} = \pm \frac{E}{\sqrt{2}} \cos\Theta \cos\chi/2 \mp \frac{E}{\sqrt{2}} \sin\Theta \sin\chi/2$$

$$\Rightarrow \quad E_1^+ + E_2^+ = \frac{2E}{\sqrt{2}} \cos\Theta \cos\chi/2$$
$$E_1^+ - E_2^+ = \frac{2E}{\sqrt{2}} \sin\Theta \sin\chi/2$$

$$\Rightarrow \quad \frac{(E_1^+ + E_2^+)^2}{2E^2\cos^2\chi/2} + \frac{(E_1^+ - E_2^+)^2}{2E^2\sin^2\chi/2} = \cos^2\Theta + \sin^2\Theta = 1$$

Similarly

$$\frac{(E_1^- - E_2^-)^2}{2E^2\cos^2\chi/2} + \frac{(E_1^- + E_2^-)^2}{2E^2\sin^2\chi/2} = 1$$

These are the equations for ellipses! $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

with semi axes

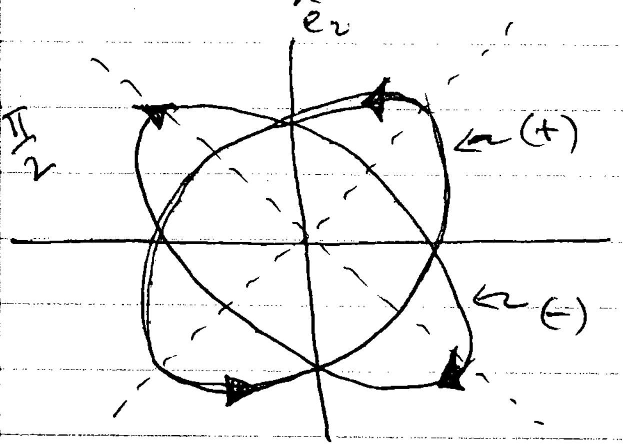
$E\cos\chi/2$ and $E\sin\chi/2$ $\quad \hat{e}_1{}' \quad \hat{e}_2{}'$

direction of the ellipse axes are $\left(\frac{\hat{e}_1 + \hat{e}_2}{\sqrt{2}}\right)$ and $\left(\frac{\hat{e}_1 - \hat{e}_2}{\sqrt{2}}\right)$

$\vec{E}\cdot\hat{e}_1{}' = \frac{E_1 + E_2}{\sqrt{2}}$, $\vec{E}\cdot\hat{e}_2{}' = \frac{E_1 - E_2}{\sqrt{2}}$

so $|\vec{E}|$ sweeps out ellipse as Φ varies.

for $|\chi| < \frac{\pi}{2}$



$\Rightarrow$ sit at position $\vec{r}$ as t varies, $|\vec{E}|$ sweeps out ellipse axis at 45° to $\hat{e}_1, \hat{e}_2$

<u>elliptically polarized wave</u>

For $0 < \chi < \pi/2$

For (+) $\vec{E}$ moves around ellipse counterclockwise (right handed

For (−) $\vec{E}$ moves around ellipse clockwise (left handed

as t increases (ie as Φ decreases)

3) Special case of (2) $\chi = \pi/2$

$\cos^2 \chi/2 = \sin^2 \chi/2 = \frac{1}{2}$ ellipse axes are equal!

$\Rightarrow \vec{E}$ sweeps out a circular path

circularly polarized waves

(+) goes counterclockwise

(−) goes clockwise

One defines circular polarization basis vectors as:

$$\hat{e}_\pm \equiv \frac{1}{\sqrt{2}}(\hat{e}_1 \pm i\hat{e}_2)$$

$$\Rightarrow \quad \hat{e}_\pm^* \cdot \hat{e}_\pm = \hat{e}_\pm \cdot \hat{e}_\mp = 1$$

$$\hat{e}_\pm \cdot \hat{e}_\mp^* = \hat{e}_\pm \cdot \hat{e}_\pm = 0$$

$$\hat{e}_\pm \cdot \hat{n} = 0$$

$$\hat{n} \times \hat{e}_\pm = i\hat{e}_\pm$$

with this notation, a circularly polarized wave is

$$\vec{E} = \mathrm{Re}\left\{ E\hat{e}_\pm e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

(+) is counterclockw

(−) is clockwise

with $\hat{n}$ pointing out

Note:

$$\frac{1}{\sqrt{2}}\left(E\,\hat{e}_+ + E\,\hat{e}_-\right) = \frac{E}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{e}_1 + \frac{i}{\sqrt{2}}\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_1 - \frac{i}{\sqrt{2}}\hat{e}_2\right)$$

$$= E\hat{e}_1$$

and $\frac{1}{\sqrt{2}i}(E\hat{e}_+ - E\hat{e}_-) = E\hat{e}_2$

Thus a linearly polarized wave can be written as a superposition of counter rotating circularly polarized waves!

general case $\qquad E_1, E_2$ complex

write $E_1\hat{e}_1 + E_2\hat{e}_2 \equiv \vec{u}\, e^{i\psi}$

where ψ is chosen so that $\vec{u}\cdot\vec{u}$ is real

(can always do this since $\vec{u}\cdot\vec{u} = (E_1^2+E_2^2)\, e^{-2i\psi}$
so 2ψ is just the phase of $E_1^2+E_2^2$)

$\vec{u}$ is complex vector $\Rightarrow$ $\vec{u} = \vec{u}_a - i\vec{u}_b$, $\vec{u}_a, \vec{u}_b$ real

since $\vec{u}\cdot\vec{u}$ is real $\Rightarrow$ $\vec{u}_a\cdot\vec{u}_b = 0$

so $\vec{u}_a \perp \vec{u}_b$ orthogonal

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{ \vec{u}\, e^{i\psi}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} \qquad \Phi \equiv k_1\hat{n}\cdot\vec{r} - \omega t$$

$$= e^{-k_2\hat{n}\cdot\vec{r}} \left\{ \vec{u}_a \cos(\Phi+\psi) + \vec{u}_b \sin(\Phi+\psi) \right\}$$

Define E_a as component of $\vec{E}$ in direction of $\vec{u}_a$
E_b as component of $\vec{E}$ in direction of $\vec{u}_b$

$$E_a = u_a \cos(\Phi+\psi)$$
$$E_b = u_b \sin(\Phi+\psi)$$

(ignore attenuation factor by either absorbing it into u_a, u_b, or consider $\vec{r}=0$)

$$\Rightarrow \left(\frac{E_a}{u_a}\right)^2 + \left(\frac{E_b}{u_b}\right)^2 = 1$$

$u_a \equiv |\vec{u}_a|$
$u_b \equiv |\vec{u}_b|$

elliptical polarization

semi axes of lengths u_a and u_b oriented in directions $\vec{u}_a$ and $\vec{u}_b$

if $u_a = \pm u_b$ then circularly polarized

Define circular polarization basis vectors

$$\hat{e}_{\pm} = \frac{1}{\sqrt{2}}\left(\hat{e}_a \pm i\hat{e}_b\right) \qquad \hat{e}_a = \frac{U_a}{|U_a|}, \quad e_b = \frac{\vec{U}_b}{|\vec{U}_b|}$$

any general $\vec{U}$ can always be written as:

$$\vec{U} = \frac{1}{\sqrt{2}}\left(U_a + U_b\right)\hat{e}_- + \frac{1}{\sqrt{2}}\left(U_a - U_b\right)\hat{e}_+$$

Thus a general elliptically polarized wave can be written as a superposition of circularly polarized waves!