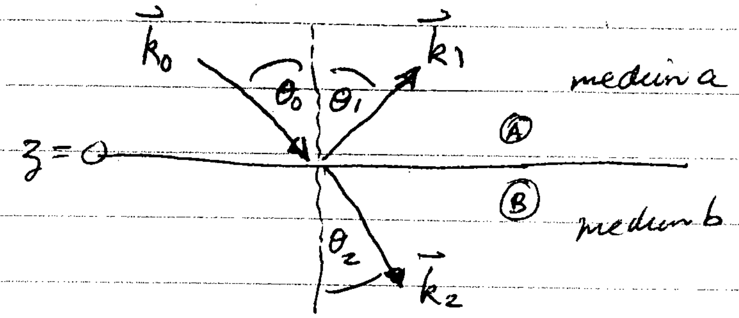
Reflection + Transmission of Waves at Interfaces



consider wave propagating from medium A into medium B.

for simplicity assume ϵ_a is real and positive, ϵ_b may be complex
μ_a and μ_b are real and constant

$\vec{k}_0$ is incident wave, θ_0 = angle of incidence
$\vec{k}_1$ is reflected wave, θ_1 = angle of reflection
$\vec{k}_2$ is the transmitted or "refracted" wave, θ_2 = angle of refraction

Let each wave be given by

$$\vec{F}_n(\vec{r},t) = \vec{F}_n\, e^{i(\vec{k}_n\cdot\vec{r} - \omega_n t)}$$

where $\vec{F}_n$ can be either $\vec{E}_n$ or $\vec{H}_n$ for the electric or magnetic component of the wave

boundary condition: tangential component $\vec{E}$ must be continuous at $z=0$. If $\hat{t}$ is a vector in xy plane, and we consider $\vec{r}=0$, then

$$\Rightarrow \quad \hat{t}\cdot\vec{E}_0\, e^{-i\omega_0 t} + \hat{t}\cdot\vec{E}_1\, e^{-i\omega_1 t} = \hat{t}\cdot\vec{E}_2\, e^{-i\omega_2 t}$$

must be true for all time. Can only happen if

$$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega}$$ all frequencies are equal

Now consider the same boundary condition for $\vec{\rho}$ a position vector in the xy plane at $z=0$. Since ω's all equal we can cancel out the common $e^{-i\omega t}$ factors to get

$$\hat{t}\cdot\vec{E}_0\, e^{i\vec{k}_0\cdot\vec{\rho}} + \hat{t}\cdot E_1\, e^{i\vec{k}_1\cdot\vec{\rho}} = \hat{t}\cdot\vec{E}_2\, e^{i\vec{k}_2\cdot\vec{\rho}}$$

this must be true for all $\vec{\rho}$. Can only happen if the projections of the $\vec{k}_n$ in the xy plane are all equal

$$\boxed{\begin{aligned} k_{0x} &= k_{1x} = k_{2x} \\ k_{0y} &= k_{1y} = k_{2y} \end{aligned}}$$

only z components $\vec{k}$ vectors can be different

choose coord system as in diagram so that all $\vec{k}$ vectors lie in the xz plane (y is out of page)

Since ϵ_a is real and positive, $\vec{k}_2$ and $\vec{k}_1$ are real vectors

$$k_{0x} = k_{1x} \Rightarrow |\vec{k}_0|\sin\theta_0 = |\vec{k}_1|\sin\theta_1$$

since $k_0^2 = \frac{\omega^2}{c^2}\sqrt{\mu_a\epsilon_a}$ and $k_1^2 = \frac{\omega^2}{c^2}\sqrt{\mu_a\epsilon_a}$

then $|\vec{k}_0| = |\vec{k}_1|$ so $\sin\theta_0 = \sin\theta_1$

$$\boxed{\theta_0 = \theta_1}$$

<u>angle of incidence = angle of reflection</u>

If ϵ_b is also real and positive (B is transparent)
then $|\vec{k}_2|$ is real

$$k_{0x} = k_{2x} \Rightarrow |\vec{k}_0| \sin\theta_0 = |\vec{k}_2| \sin\theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2} \sqrt{\mu_b \epsilon_b}$$

$$\Rightarrow \sqrt{\mu_a \epsilon_a} \sin\theta_0 = \sqrt{\mu_b \epsilon_b} \sin\theta_2$$

in terms of index of refraction $n \equiv \frac{kc}{\omega} = \frac{\omega\sqrt{\mu\epsilon}}{c}\frac{c}{\omega}$

$$n = \sqrt{\mu\epsilon}$$

$$\Rightarrow n_a \sin\theta_0 = n_b \sin\theta_2$$

$$\boxed{\frac{\sin\theta_2}{\sin\theta_0} = \frac{n_a}{n_b}}$$

Snell's Law
true for all types of waves, not just EM wave.

If $n_a > n_b$ then $\theta_2 > \theta_0$
In this case, when θ_0 is too large, we will have
$\frac{n_a}{n_b} \sin\theta_0 > 1$ and there will be no solution for θ_2
$\Rightarrow$ no transmitted wave

This is "total internal reflection" – wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

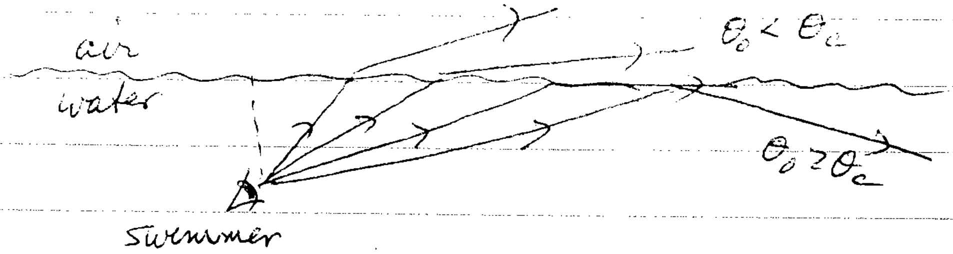
$$\frac{n_a}{n_b} \sin\theta_c = 1 \quad , \quad \boxed{\theta_c = \arcsin\left(\frac{n_b}{n_a}\right)}$$

$$\epsilon \sim 1 + 4\pi N\alpha$$ ← density

Since $n = \sqrt{\mu\epsilon}$ and ϵ grows with density of the material, one usually has total internal reflection when one goes from a denser to a less dense medium.

Examples: diamonds sparkle due to total internal reflection. Diamonds have large $n \Rightarrow$ small θ_c $\Rightarrow$ light bounces around inside many times before it can exit

Can also see total internal reflection when swimming under water



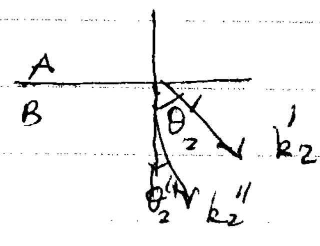
<u>More general case</u> $\sqrt{\epsilon_b}$ is complex so k_2 is complex

$$\vec{k}_2 = \underbrace{\vec{k}_2'}_{\text{real part}} + i\underbrace{\vec{k}_2''}_{\text{imaginary part}} \qquad k_2' = |\vec{k}_2'| \qquad k_2'' = |\vec{k}_2''|$$

Note $\vec{k}_2'$ and $\vec{k}_2''$ need not be in the same direction!

Condition $k_{0x} = k_{2x} \Rightarrow \begin{cases} k_{0x} = k_{2x}' \\ 0 = k_{2x}'' \end{cases}$ equate real and imaginary parts

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$
$$0 = k_2'' \sin\theta_2''$$



$\Rightarrow \theta_2'' = 0$
$\vec{k}_2'' = k_2'' \hat{z}$ } attenuation factor for the transmitted wave is $e^{-k_2'' z}$

$\Rightarrow$ planes of constant amplitude are parallel to the interface no matter what the angle of incidence θ_0

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$

$$k_0 = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a} = \frac{\omega}{c} n_a$$

planes of constant phase are $\perp$ to $\vec{k}_2'$

dispersion relation

$$k_2^2 = \vec{k}_2 \cdot \vec{k}_2 = (k_2')^2 - (k_2'')^2 + 2i\, \vec{k}_2' \cdot \vec{k}_2'' = \frac{\omega^2}{c^2}\mu_b \epsilon_b$$

$$\vec{k}_2' \cdot \vec{k}_2'' = k_2' k_2'' \cos\theta_2'$$

equate real and imaginary parts

$$(k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$\epsilon_b = \epsilon_{b1} + i\epsilon_{b2}$

real

$$2 k_2' k_2'' \cos\theta_2' = \frac{\omega^2}{c^2}\mu_b \epsilon_{b2}$$

solve

$$(k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$(k_2')^2 = \left(\frac{\frac{\omega^2}{c^2}\mu_b \epsilon_{b2}}{2k_2' \cos\theta_2'}\right)^2 + \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$(k_2')^4 - \frac{\omega^2}{c^2}\mu_b\epsilon_{b1}(k_2')^2 - \frac{\omega^4}{c^4}\frac{\mu_b^2\epsilon_{b2}^2}{4\cos^2\theta_2'} = 0$$

quadratic formula

$$(k_2')^2 = \frac{\omega^2\mu_b\epsilon_{b1}}{c^2\,2} + \sqrt{\frac{\omega^4\mu_b^2\epsilon_{b1}^2}{c^2\,4} + \frac{\omega^4\mu_b^2\epsilon_{b2}^2}{c^2\,4\cos^2\theta_2'}}$$

$$\boxed{k_2' = \frac{\omega}{c}\sqrt{\mu_b}\left[\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\,\right]^{1/2}}$$

and

$$(k_2'')^2 = (k_2')^2 - \frac{\omega^2}{c^2}\mu_b\epsilon_{b1}$$

$$\boxed{k_2'' = \frac{\omega}{c}\sqrt{\mu_b}\left[-\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\,\right]^{1/2}}$$

Note, these reduce to what we had earlier for a plane wave, if we take $\theta_2' = 0$

Both k_2' and k_2'' depend on angle of refraction θ_2'

Finally: $k_2'\sin\theta_2' = \frac{\omega}{c}n_a\sin\theta_0$

$$\Rightarrow \boxed{n_a\sin\theta_0 = \sqrt{\mu_b\epsilon_{b1}}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\epsilon_{b2}^2}{\epsilon_{b1}^2\cos^2\theta_2'}}\,\right]^{1/2}\sin\theta_2'}$$

determines θ_2' in terms of given θ_0

Cases

① for a nearly transparent material with $\epsilon_{b2} \ll \epsilon_{b1}$

define $n_b = \sqrt{\mu_b\epsilon_{b1}}$ index of refraction

$$n_a \sin\theta_0 = n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{4\epsilon_{b1}^2 \cos^2\theta_2'}\right]^{1/2}$$

$$\cong n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \cos^2\theta_2'}\right]$$

↑ small correction to Snell's law

for $\frac{\epsilon_{b2}}{\epsilon_{b1}} \ll 1$ can solve iteratively

to lowest order: $n_a \sin\theta_0 \simeq n_b \sin\theta_2'$

$$\Rightarrow \cos^2\theta_2' = 1 - \sin^2\theta_2' = 1 - \left(\frac{n_a}{n_b}\sin\theta_0\right)^2$$

so to next order

$$n_a \sin\theta_0 \simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2\left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]$$

or
$$\sin\theta_2' \simeq \frac{n_a}{n_b}\sin\theta_0 \frac{1}{\left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2\left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]}$$

$$\leq \frac{n_a}{n_b}\sin\theta_0$$

result is that θ_2' is <u>smaller</u> than Snell's law would predict.

(2) for a good conductor, or absorbing region of a dielectric, $\epsilon_{b2} \gg \epsilon_{b1}$

to lowest order

$$n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}} \left[\frac{1}{2} \frac{\epsilon_{b2}}{\epsilon_{b1} \cos\theta_2'} \right]^{1/2} \sin\theta_2'$$

$$n_a \sin\theta_0 = \sqrt{\frac{\mu_b \epsilon_{b2}}{2}} \frac{\sin\theta_2'}{\sqrt{\cos\theta_2'}}$$

very differ[ent] from Snell's Law!

Snell's Law only holds if both media are transparent