

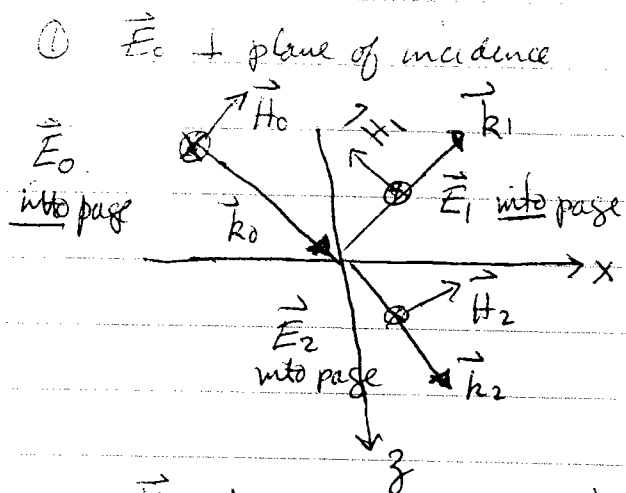
Reflection coefficients

Now we compute the amplitude of the reflected wave to determine how much of incident wave is reflected and how much is transmitted.

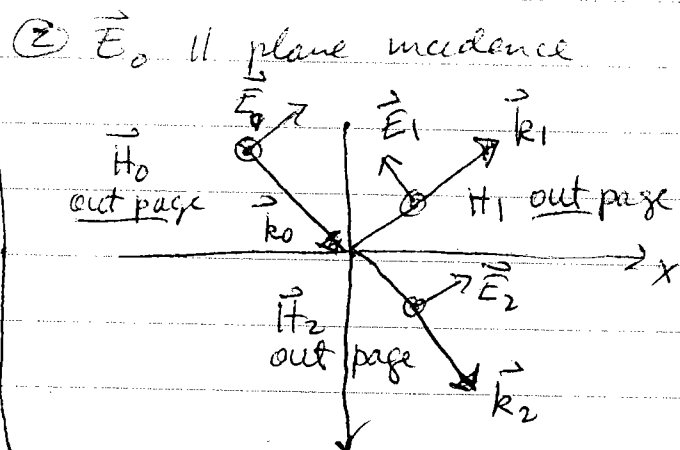
Consider two cases ① $\vec{E}_0$ is $\perp$ plane of incidence

② $\vec{E}_0$ lies in the plane of incidence

"plane of incidence" is the plane spanned by the wave vector $\vec{k}_0$ and the normal to the interface - in our case it is the xz plane



$\Rightarrow \vec{H}_0$ in plane of incidence
all $\vec{E}$'s are in $\hat{y}$ direction



$\Rightarrow \vec{H}_0 \perp$ plane of incidence
all the $\vec{H}$'s are in y direction

continuity of y components

$$1) E_0 + E_1 = E_2$$

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continuity of x components

$$H_{0x} + H_{1x} = H_{2x}$$

$$E_{0x} + E_{1x} = E_{2x}$$

Faraday

$$\vec{\nabla} \times \vec{H} = \vec{\nabla} \times \vec{E} \Rightarrow H_x = \frac{k_z c}{\omega \mu} E_y$$

Ampere

$$-\omega \epsilon \vec{E} = \vec{\nabla} \times \vec{H} \Rightarrow E_x = -\frac{k_z c}{\omega \epsilon} H_y$$

⇒

$$2) \frac{k_{0z}}{\mu_a} (E_0 - E_1) = \frac{k_{2z}}{\mu_b} E_2$$

Solve (1) and (2) for
 E_1 and E_2 in terms of E_0

$$E_1 = \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} E_0$$

$$E_2 = \frac{2\mu_b k_{0z}}{\mu_a k_{2z} + \mu_b k_{0z}} E_0$$

$$2) \frac{k_{0z}}{\epsilon_a} (H_0 - H_1) = \frac{k_{2z}}{\epsilon_b} H_2$$

Solve (1) and (2) for
 H_1 and H_2 in terms of H_0

$$H_1 = \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} H_0$$

$$H_2 = \frac{2\epsilon_b k_{0z}}{\epsilon_a k_{2z} + \epsilon_b k_{0z}} H_0$$

Define reflection coefficient in terms of the transported energy
 $R = \frac{|E_1|^2}{|E_0|^2} = \frac{|H_1|^2}{|H_0|^2}$

Reflection coefficients

① $\vec{E}_0 \perp$ plane incidence

$$R_{\perp} = \frac{|E_1|^2}{|E_0|^2} = \left| \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} \right|^2$$

② $\vec{E}_0 \parallel$ plane incidence

$$R_{\parallel} = \frac{|H_1|^2}{|H_0|^2} = \left| \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} \right|^2$$

Note: above are correct for an arbitrary medium B

i) Consider region of "total reflection"

$$\Rightarrow \left. \begin{array}{l} \text{Im } \epsilon_b = \epsilon_{b2} \approx 0 \\ \text{Re } \epsilon_b = \epsilon_{b1} < 0 \end{array} \right\} \Rightarrow \vec{k}_2 = i \vec{K}_2 \quad \text{where } \vec{K}_2 \text{ is real} \\ \text{ie } \vec{K}_2 \text{ pure imaginary}$$

$$\Rightarrow R_{\perp} = \left| \frac{\mu_b k_{0z} - i \mu_a K_{2z}}{\mu_b k_{0z} + i \mu_a K_{2z}} \right|^2$$

$$R_{\parallel} = \left| \frac{\epsilon_b k_{0z} - i \epsilon_a K_{2z}}{\epsilon_b k_{0z} + i \epsilon_a K_{2z}} \right|^2$$

both are of the form $\left| \frac{a - ib}{a + ib} \right|^2 = 1$ when a, b real

$$\Rightarrow R_{\perp} = R_{\parallel} = 1$$

confirms that the material is completely reflecting

ii) Next consider when medium B is transparent

ϵ_b is real and $\epsilon_b > 0$

$$k_{0z} = \frac{\omega}{c} \sqrt{\mu_a \epsilon_a} \cos \theta_0 = \frac{\omega}{c} \mu_a \cos \theta_0$$

$$k_{2z} = \frac{\omega}{c} \sqrt{\mu_b \epsilon_b} \cos \theta_2 = \frac{\omega}{c} \mu_b \cos \theta_2$$

Snell's law holds so $\mu_a \sin \theta_0 = \mu_b \sin \theta_2$

can write $R_{\perp}$ and $R_{\parallel}$ as functions of θ_0
for simplicity take $\mu_a = \mu_b = 1$

$$\textcircled{1} R_{\perp} = \left(\frac{m_a \cos \theta_0 - m_b \cos \theta_2}{m_a \cos \theta_0 + m_b \cos \theta_2} \right)^2 = \left(\frac{\cos \theta_0 - \left(\frac{\sin \theta_0}{\sin \theta_2} \right) \cos \theta_2}{\cos \theta_0 + \left(\frac{\sin \theta_0}{\sin \theta_2} \right) \cos \theta_2} \right)^2$$

$$= \left(\frac{\sin \theta_2 \cos \theta_0 - \sin \theta_0 \cos \theta_2}{\sin \theta_2 \cos \theta_0 + \sin \theta_0 \cos \theta_2} \right)^2$$

$$R_{\perp} = \left(\frac{\sin(\theta_0 - \theta_2)}{\sin(\theta_0 + \theta_2)} \right)^2$$

for $\theta_0 = 0$, i.e. normal incidence, $\theta_2 = 0$

$$\Rightarrow R_{\perp} = \left(\frac{m_a - m_b}{m_a + m_b} \right)^2 \quad \text{if } m_a = m_b, \text{ no reflection!}$$

(not surprising!)

$$\textcircled{2} R_{\parallel} = \left(\frac{\epsilon_b m_a \cos \theta_0 - \epsilon_a m_b \cos \theta_2}{\epsilon_b m_a \cos \theta_0 + \epsilon_a m_b \cos \theta_2} \right)^2$$

use $\sqrt{\epsilon_b} = m_b$
 $\sqrt{\epsilon_a} = m_a$

$$= \left(\frac{m_b \cos \theta_0 - m_a \cos \theta_2}{m_b \cos \theta_0 + m_a \cos \theta_2} \right)^2$$

$$= \left(\frac{\cos \theta_0 - \left(\frac{\sin \theta_2}{\sin \theta_0} \right) \cos \theta_2}{\cos \theta_0 + \left(\frac{\sin \theta_2}{\sin \theta_0} \right) \cos \theta_2} \right)^2$$

$$= \left(\frac{\sin \theta_0 \cos \theta_0 - \sin \theta_2 \cos \theta_2}{\sin \theta_0 \cos \theta_0 + \sin \theta_2 \cos \theta_2} \right)^2$$

$$R_{\parallel} = \left(\frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)} \right)^2 \quad \leftarrow \text{after some algebra!}$$

for $\theta_0 = 0$, then $\theta_2 = 0$

$$R_{||} = \left(\frac{\epsilon_b m_a - \epsilon_a m_b}{\epsilon_b m_a + \epsilon_a m_b} \right)^2 = \left(\frac{m_b - m_a}{m_b + m_a} \right)^2 \text{ same as } R_{\perp}$$

So for $\theta_0 = 0$, $R_{||} = R_{\perp}$ - this must be so since for $\theta_0 = 0$ there is no distinction between the $\perp$ and $\parallel$ cases

If $m_b = m_a$, $R_{\perp} = R_{||} = 0$ no reflective wave

When $\theta_0 + \theta_2 = \pi/2$, then $\tan(\theta_0 + \theta_2) \rightarrow \infty$
and $R_{||} = 0$

This occurs at an angle of incidence known as Brewster's angle θ_B , determined by

$$\begin{array}{ccc} m_a \sin \theta_B & = & m_b \sin \left(\frac{\pi}{2} - \theta_B \right) = m_b \cos \theta_B \\ \uparrow & & \uparrow \\ \theta_0 & & \theta_2 \end{array}$$

$$\Rightarrow \boxed{\tan \theta_B = \frac{m_b}{m_a}}$$

For incident wave at θ_B , reflected wave always has $\vec{E}_r \perp$ plane of incidence, since $R_{||} = 0$. If incoming wave has $\vec{E}_0 \parallel$ plane of incidence, then it gets completely transmitted. If $\vec{E}_0$ in general direction, reflected wave is always linearly polarized with $\vec{E}_r \perp$ plane of incidence. - This is one method to create polarized light wave

Kramers - Kronig Relation

We saw that $\vec{P}_\omega = \alpha(\omega) \vec{E}_\omega$

Causal response is $\tilde{\alpha}(t) = 0$ for $t < 0$

$\Rightarrow \alpha(\omega)$ has no poles in upper half of complex ω plane (UHP)

For any complex $\bar{\omega}$ in upper half of complex ω plane,

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i} \oint \frac{\alpha(\omega')}{\omega' - \bar{\omega}} d\omega' \quad \text{since no poles of } \alpha \text{ in UHP}$$

$\Rightarrow$

contour along real axis, closed at infinity in UHP. The closing ~~top~~ semicircle at infinity gives no contribution assuming $\alpha(\omega)$ decays quickly enough as $|\omega| \rightarrow \infty$

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \bar{\omega}}$$

Now consider $\bar{\omega} = \omega + i\delta$ where ω and δ are real and $\delta \rightarrow 0$

$$\alpha(\omega) = \lim_{\delta \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \omega - i\delta}$$

$$\text{Now } \frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + i\pi\delta(\omega' - \omega)$$

$\uparrow$ principal part

$$\Rightarrow \alpha(\omega) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{\alpha(\omega') d\omega'}{\omega' - \omega}$$

$$\begin{aligned} \Rightarrow \left. \begin{aligned} \operatorname{Re} \alpha(\omega) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} \alpha(\omega') d\omega'}{\omega' - \omega} \\ \operatorname{Im} \alpha(\omega) &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Re} \alpha(\omega') d\omega'}{\omega' - \omega} \end{aligned} \right\} \begin{array}{l} \text{Kramers} \\ \text{Kronig} \\ \text{relations} \end{array}$$

If know $\operatorname{Re} \alpha$ or $\operatorname{Im} \alpha$ can reconstruct full complex α

True for any causal response function