For arbitrary charge distributions — not pure harmonic

For $\vec{p}_\omega e^{-i\omega t}$ pure harmonic oscillation, we found the radiated fields in electric dipole approx are

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}, \quad \vec{B} = \vec{B}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega = -k^2 \frac{e^{ikr}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega) = -\frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$\vec{B}_\omega = k^2 \frac{e^{ikr}}{r} (\hat{r} \times \vec{p}_\omega) = \frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} (\hat{r} \times \vec{p}_\omega)$$

as $k = \frac{\omega}{c}$

For an arbitrarily time varying charge distribution with electric dipole moment

$$\vec{p}(t) = \int \frac{d\omega}{2\pi} \vec{p}_\omega e^{-i\omega t}$$

then solution for fields given by superposition

$$\vec{E}(\vec{r},t) = \int \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$= -\int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2}\right) \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$= \frac{-1}{c^2 r} \hat{r} \times \left[\hat{r} \times \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega \, \omega^2\right]$$

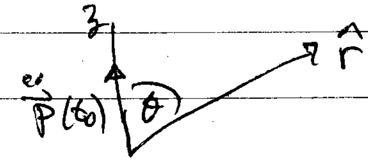
$$= \frac{1}{c^2 r} \hat{r} \times \left[\hat{r} \times \frac{\partial^2}{\partial t^2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega\right]$$

$$\boxed{\vec{E}(\vec{r},t) = \frac{1}{c^2 r}\,\hat{r}\times\left[\hat{r}\times\ddot{\vec{p}}(t-r/c)\right]} \qquad \ddot{\vec{p}} = \frac{d^2\vec{p}}{dt^2}$$

define $t_0 \equiv t - r/c$ = "retarded time"

in spherical coords, if $\ddot{\vec{p}}(t_0)$ is along $\hat{z}$

$$\vec{E}(\vec{r},t) = \frac{\ddot{p}(t_0)}{c^2 r}\sin\theta\,\hat{\theta}$$



Similarly

$$\vec{B}(\vec{r},t) = \int\frac{d\omega}{2\pi}\,\vec{B}_\omega\,e^{-i\omega t}$$

$$= \int\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-r/c)}}{r}\left(\frac{\omega^2}{c^2}\right)(\hat{r}\times\vec{p}_\omega)$$

$$= \frac{-1}{c^2 r}\,\hat{r}\times\frac{\partial^2}{\partial t^2}\int\frac{d\omega}{2\pi}\,e^{-i\omega(t-r/c)}\,\vec{p}_\omega$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{-1}{c^2 r}\,\hat{r}\times\ddot{\vec{p}}(t_0)}$$

$$\vec{B}(\vec{r},t) = \frac{\ddot{p}(t_0)}{c^2 r}\sin\theta\,\hat{\varphi} \qquad \text{in spherical coords}$$

<u>Poynting vector</u>

$$\vec{S} = \frac{c}{4\pi}\,\vec{E}\times\vec{B} = \frac{c}{4\pi}\left(\frac{1}{c^2 r}\right)^2\left[\ddot{p}(t_0)\right]^2\sin^2\theta\,\hat{r}$$

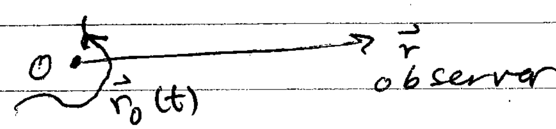
Total power radiated through a sphere of radius r is

$$\mathcal{P} = \oint da\, \hat{r}\cdot\vec{S} = 2\pi\int_0^\pi d\theta \sin\theta\, r^2\, \hat{r}\cdot\vec{S}$$

$$= \frac{[\ddot{p}(t_0)]^2}{2c^3} \underbrace{\int_0^\pi d\theta \sin^3\theta}_{4/3}$$

$$\boxed{\mathcal{P} = \frac{2\,[\ddot{p}(t_0)]^2}{3c^3}}$$

For a point charge moving along a trajectory $\vec{r}_0(t)$



$\vec{p}(t) = q\,\vec{r}_0(t)$

$\ddot{\vec{p}}(t) = q\,\ddot{\vec{r}}_0(t) = q\,\vec{a}(t)$

↑ acceleration

$$\boxed{\mathcal{P} = \frac{2}{3}\frac{q^2 a^2(t_0)}{c^3}}$$ ← total power passing through a sphere of radius r at time t is due to acceleration at retarded time $t_0 = t - r/c$

Larmor's formular

power radiated ~ (acceleration)^2

Larmor's formular above only holds in the non-relativistic limit since it is based on the electric dipole approx.

To go beyond non-relativistic limit, one can do the following:

1) transform to a new inertial frame of reference in which the charge is instantaneously at rest
2) apply non-relativistic Larmor formula in this frame
3) transform back to "lab" frame

Back to non-relativistic case

radiation fields from moving point charge

$$\vec{E}(\vec{r},t) = \frac{q}{c^2 r}\,\hat{r}\times(\hat{r}\times\vec{a}(t_0))$$

$$\vec{B}(\vec{r},t) = -\frac{q}{c^2 r}\,(\hat{r}\times\vec{a}(t_0))$$

Poynting vector

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = -\frac{c}{4\pi}\left(\frac{q}{c^2 r}\right)^2\left[\hat{r}\times(\hat{r}\times\vec{a})\right]\times\left[\hat{r}\times\vec{a}\right]$$

$$= \frac{c}{4\pi}\left(\frac{q}{c^2 r}\right)^2\left\{\hat{r}\,(\hat{r}\times\vec{a})^2 - (\hat{r}\times\vec{a})\left[\hat{r}\cdot(\hat{r}\times\vec{a})\right]\right\}$$

$$= \frac{q^2}{4\pi c^3}\frac{\left[\hat{r}\times\vec{a}(t_0)\right]^2}{r^2}\,\hat{r}$$

$$\boxed{\vec{S} = \frac{q^2}{4\pi c^3 r^2}\left(a^2 - (\hat{r}\cdot\vec{a})^2\right)\hat{r}}$$

$\vec{a}$ evaluated at t_0

for $\vec{a} = a\hat{z}$

$$\boxed{= \frac{q^2}{4\pi c^3}\frac{a^2(t_0)}{r^2}\sin^2\theta\,\hat{r}}$$

Special Relativity

1) Speed of light is constant in all inertial frames of reference
2) Physical laws must look the same in all inertial frames of refence – there is no experiment that can determine the "absolute" velocity of any inertial frame

$\Rightarrow$ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.
Equation of wavefront is $r^2 - c^2t^2 = 0$

$\Rightarrow$ (x, y, z, t) coords in one inertial frame K
(x', y', z', t') coords in another inertial frame K' that moves with velocity $\vec{v} = v\hat{x}$ with respect to K.
What is the transformation that relates coords in K' to coords in K

$y = y'$, $z = z'$

$$\Rightarrow \quad c^2t^2 - x^2 = c^2t'^2 - x'^2$$

$$\Rightarrow \quad \frac{(ct+x)}{(ct'+x')}\frac{(ct-x)}{(ct'-x')} = 1$$

Expect transformation to be <u>linear</u>

$$\Rightarrow \quad ct' + x' = (ct + x)\, f$$
$$ct' - x' = (ct - x)\, f^{-1}$$

for some constant f. Write $f = e^{-y}$ y is <u>rapidity</u>

Solve for ct' and x' in terms of ct and x

$$ct' = ct\left(\frac{e^y + e^{-y}}{2}\right) - x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$x' = -ct\left(\frac{e^y - e^{-y}}{2}\right) + x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$ct' = ct \cosh y - x \sinh y$$
$$x' = -ct \sinh y + x \cosh y$$

<u>meaning of parameter y</u>

the origin of K (at $x=0$) has trajectory $x' = -vt'$ in K'

$$\Rightarrow \quad \frac{x'}{t'} = -v$$

from transformation above, with $x=0$, we get

$$\frac{x'}{ct'} = \frac{-ct \sinh y}{ct \cosh y} = -\tanh y$$

so $\quad \frac{v}{c} = \tanh y$

$$\Rightarrow \quad \cosh y = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma$$

$$\sinh y = \left(\frac{v}{c}\right)\gamma$$

<u>Lorentz Transformation</u>
$$\begin{cases} ct' = \gamma ct - \gamma\left(\frac{v}{c}\right)x \\ x' = -\gamma\left(\frac{v}{c}\right)ct + \gamma x \end{cases}$$

Inverse transform obtained by taking $v \to -v$ in above

$$\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right) x' \\ x = \gamma\left(\frac{v}{c}\right) ct' + \gamma x' \end{cases}$$

4 - vectors

4-position: $x_\mu = (x_1, x_2, x_3, ict)$ $\qquad x_4 \equiv it$

$$x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu^2 = r^2 - c^2 t^2$$

Lorentz invariant scalar – has same value in all inertial frames

Lorentz transf is

$$\left.\begin{aligned} x_1' &= \gamma\left(x_1 + i\left(\tfrac{v}{c}\right) x_4\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma\left(x_4 - i\left(\tfrac{v}{c}\right) x_1\right) \end{aligned}\right\}$$

linear transf, can be represented by a matrix

or $\quad x_\mu' = a_{\mu\nu}(L)\, x_\nu$

$\uparrow$ matrix of Lorentz transformation L

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse: $\quad x_\mu = a_{\mu\nu}(L^{-1})\, x_\nu'$

$a_{\mu\nu}(L^{-1})$ is given by taking $v \to -v$ in $a_{\mu\nu}(L)$

we see $\quad a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose

More generally

Since x_μ^2 is Lorentz invariant scalar,

$$x'^2_\mu = a_{\mu\nu}(L)\, a_{\mu\lambda}(L)\, x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow \quad a_{\mu\nu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\nu\mu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\mu\nu} = a^{-1}_{\mu\nu}(L)$$ transpose = inverse

$a_{\mu\nu}$ is 4×4 orthogonal matrix

If L_1 is a Lorentz transf from K to K'
L_2 is a Lorentz transf from K' to K''
Then the Lorentz transf from K to K'' is given by the matrix

$$a(L_2 L_1) = a(L_2)\, a(L_1)$$

$$\Rightarrow \quad a^{-1}(L) = a(L^{-1})$$

$$dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2$$ Lorentz invariant scalar

$$ds^2 = dt^2\left[1 - \frac{1}{c^2}\left(\frac{dx_1}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_2}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_3}{dt}\right)^2\right]$$

$$ds^2 = \frac{dt^2}{\gamma^2}$$

$$\boxed{ds = \frac{dt}{\gamma}}$$ proper time interval

A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

4-velocity $\quad u_\mu \equiv \frac{dx_\mu}{ds} \equiv \dot{x}_\mu$

$$= \gamma \frac{dx_\mu}{dt}$$

space components $\vec{u} = \gamma \vec{v}$

$u_4 = ic\gamma$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2\gamma^2 = \gamma^2 (v^2 - c^2)$$

$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$$

4-acceleration $\quad a_\mu \equiv \frac{du_\mu}{ds} = \gamma \frac{du_\mu}{dt}$

4-gradient $\quad \frac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, -\frac{i}{c}\frac{\partial}{\partial t} \right)$

proof $\frac{\partial}{\partial x_\mu}$ is a 4-vector

$$\frac{\partial}{\partial x'_\mu} = \frac{\partial x_\lambda}{\partial x'_\mu} \frac{\partial}{\partial x_\lambda}$$

but $\frac{\partial x_\lambda}{\partial x'_\mu} = a_{\lambda\mu}(L^{-1})$

$= a_{\mu\lambda}(L)$

$$= a_{\mu\lambda}(L) \frac{\partial}{\partial x_\lambda}$$

so transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu}\right)^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

wave equation operator!

inner products

If U_μ and V_μ are 4-vectors, then

$U_\mu V_\mu$ is Lorentz invarient scalar