

Electro magnetic Potentials & Gauge Invariance

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Consider first statics.

Electro statics

$$\vec{\nabla} \times \vec{E} = 0 \quad \text{since } \frac{\partial \vec{B}}{\partial t} = 0 \text{ for statics}$$

From vector calculus we know that if the curl of a vector is ~~also~~ everywhere zero, then we can always write that vector field as the gradient of some scalar function ϕ

$$\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla}\phi) = 0$$

ϕ is electrostatic potential

Gauss Law becomes

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot (\vec{\nabla}\phi) = -\nabla^2\phi = 4\pi\rho$$

$$\boxed{\nabla^2\phi = -4\pi\rho} \quad \text{Poisson's Equation}$$

In regions where $\rho = 0$, we have

$$\nabla^2\phi = 0 \quad \text{Laplace's Equation}$$

In our discussion of Coulomb's Law we saw that the electric field from a distribution of localized charges was

$$\begin{aligned}\vec{E}(\vec{r}) &= \int d^3r' \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \\ &= -\vec{\nabla} \left[\int d^3r' \rho(\vec{r}') \frac{1}{|\vec{r}-\vec{r}'|} \right] = -\vec{\nabla} \phi\end{aligned}$$

We therefore see that the solution to Poisson's eqn for a localized charge distribution ρ (with $\vec{E} = 0$ as $\vec{r} \rightarrow \infty$) is

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

We will soon spend a fair amount of time learning new ways to solve $\nabla^2 \phi = \rho$, both for arbitrary ρ where we want an approx to the above integral (multipole expansion), and for cases where ϕ or $\vec{\nabla} \phi$ are predetermined on the surfaces of specified regions of space, such as conducting surfaces (boundary value problems).

Magnetostatics

$$\vec{\nabla} \cdot \vec{B} = 0$$

From vector calculus we know that if the divergence of a vector field vanishes everywhere, then it can always be written as the curl of another vector field $\vec{A}$

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$\vec{A}$ is the magnetic vector potential

This remains true in general - not just in magnetostatics

Ampere's Law becomes

$$\vec{\nabla} \times \vec{B} = \frac{4\pi \vec{j}}{c} \quad (\text{in magnetostatics } \frac{\partial \vec{E}}{\partial t} = 0)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{4\pi \vec{j}}{c}$$

$$\Rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \frac{4\pi \vec{j}}{c}$$

magnetostatic gauge invariance

There are many possible vector potentials $\vec{A}$ that result in the same $\vec{B}$. If $\vec{A}$ is such that $\vec{\nabla} \times \vec{A} = \vec{B}$, then $\vec{A}' = \vec{A} + \vec{\nabla} \chi$ also gives $\vec{\nabla} \times \vec{A}' = \vec{B}$, since $\vec{\nabla} \times \vec{\nabla} \chi = 0$ for any scalar function $\chi(r)$.

Therefore we can always choose to represent $\vec{B}$ by a vector potential $\vec{A}$ such that $\vec{\nabla} \cdot \vec{A} = 0$.

proof: Suppose we had $\vec{B} = \vec{\nabla} \times \vec{A}$ for some $\vec{A}$ with $\vec{\nabla} \cdot \vec{A} = D(\vec{r}) \neq 0$. Construct an $\vec{A}' = \vec{A} + \vec{\nabla} \chi$ with χ chosen as follows:

$$\vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \nabla^2 \chi = 0 \Rightarrow \nabla^2 \chi = -\vec{\nabla} \cdot \vec{A} = D$$

Solve for χ , for example

$$\chi(\vec{r}) = \frac{\int d^3 r' D(\vec{r}')}{4\pi |\vec{r} - \vec{r}'|}$$

we thus have constructed an $\vec{A}'$ such that $\vec{\nabla} \times \vec{A}' = \vec{B}$ and $\vec{\nabla} \cdot \vec{A}' = 0$.

This freedom to choose various $\vec{A}$'s that give the same $\vec{B}$ is called gauge invariance. Imposing a particular additional constraint on $\vec{A}$ that removes this freedom is called fixing the gauge. The choice $\vec{\nabla} \cdot \vec{A} = 0$ is usually known as the Coulomb gauge (or sometimes the Landau gauge). Going from $\vec{A}$ to $\vec{A}' = \vec{A} + \vec{\nabla} \chi$ is called making a gauge transformation.

"Working in the Coulomb gauge" with $\vec{\nabla} \cdot \vec{A} = 0$, Ampere's Law becomes

$$\boxed{\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{j}} \quad \text{Poisson's Equ.}$$

For a localized current density

$$\vec{A}(\vec{r}) = \frac{1}{c} \int d^3 r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

Back to dynamics

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \text{ remains true}$$

But now instead of $\vec{\nabla} \times \vec{E} = 0$ we have

$$\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial (\vec{\nabla} \times \vec{A})}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$\Rightarrow$ there exists a scalar potential ϕ such that

$$\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \phi \quad \text{or} \quad \boxed{\vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}}$$

Gauss's law for electric field now becomes

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho = -\nabla^2 \phi - \frac{1}{c} \frac{\partial (\vec{\nabla} \cdot \vec{A})}{\partial t} = 4\pi\rho$$

$$\boxed{\nabla^2 \phi + \frac{1}{c} \frac{\partial (\vec{\nabla} \cdot \vec{A})}{\partial t} = -4\pi\rho} \quad \text{Gauss law in terms of electromagnetic potentials}$$

Ampere's law becomes

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{\nabla} \phi}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$-\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \frac{4\pi}{c} \vec{j} - \frac{1}{c} \frac{\partial}{\partial t} \left(\vec{\nabla} \phi + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right)$$

or
$$-\nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c} \vec{j} - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right)$$

Gauge invariance

As before, we can always construct $\vec{A}' = \vec{A} + \vec{\nabla} \chi$, for any scalar function χ , that gives the same $\vec{B}$. But since $\vec{A}$ now also enters expression for $\vec{E}$, we need to make sure that if we change $\vec{A}$ to $\vec{A}'$, we must make some corresponding change ϕ to ϕ' so that $\vec{E}$ does not change.

$$\left[\begin{array}{l} \vec{A}' = \vec{A} + \vec{\nabla} \chi \\ \phi' = \phi - \frac{1}{c} \frac{\partial \chi}{\partial t} \end{array} \right] \quad \text{gauge transformation}$$

For any scalar χ , the above $\vec{A}'$ and ϕ' give the same values of $\vec{E}$ and $\vec{B}$ as $\vec{A}$ and ϕ .

Proof:

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} \chi = \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\begin{aligned} \left(-\vec{\nabla} \phi' - \frac{1}{c} \frac{\partial \vec{A}'}{\partial t} \right) &= -\vec{\nabla} \phi + \frac{1}{c} \vec{\nabla} \frac{\partial \chi}{\partial t} - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \chi \\ &= \left(-\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = \vec{E} \end{aligned}$$

As before, we can fix the gauge by imposing some additional constraint on $\vec{A}$ and ϕ . There are two popular choices:

1) Lorentz Gauge

gauge constraint: require $\frac{1}{c} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$

Then Gauss' Law becomes

$$\nabla^2 \phi + \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -4\pi \rho$$

$$\Rightarrow \nabla^2 \phi - \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \phi}{\partial t} \right) = -4\pi \rho$$

$$\boxed{\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho}$$

Ampere's Law becomes

$$-\nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c} \vec{j} - \vec{\nabla} \left(\underbrace{\vec{\nabla} \cdot \vec{A}}_0 + \frac{1}{c} \frac{\partial \phi}{\partial t} \right)$$

$$\boxed{\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{j}}$$

The combination $-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \equiv \square^2$
is the wave equation operator.

In Lorentz gauge, $\vec{A}$ and ϕ satisfy the inhomogeneous wave equations:

$$\boxed{\begin{aligned} \square^2 \vec{A} &= \frac{4\pi}{c} \vec{j} \\ \square^2 \phi &= 4\pi \rho \end{aligned}}$$

when $\vec{j}=0, \rho=0$
electromagnetic waves
are solution!

Note: Lorentz gauge condition does not uniquely determine $\vec{A}$ and ϕ . If one constructs $\vec{A}$ and ϕ obeying Lorentz gauge condition, and then constructs

$$\begin{aligned}\vec{A}' &= \vec{A} + \vec{\nabla} \chi \\ \phi' &= \phi - \frac{1}{c} \frac{\partial \chi}{\partial t}\end{aligned}$$

then $\vec{A}'$ and ϕ' will also be in Lorentz gauge provided $\square^2 \chi = 0$ (proof left to reader)

2) Coulomb Gauge

gauge constraint: require $\vec{\nabla} \cdot \vec{A} = 0$
if $\vec{A}$ is in the Coulomb Gauge, then
 $\vec{A}' = \vec{A} + \vec{\nabla} \chi$ will also be in Coulomb gauge
provided $\nabla^2 \chi = 0$.

Then Gauss' law becomes

$$\nabla^2 \phi + \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -4\pi \rho$$

$$\Rightarrow \boxed{\nabla^2 \phi = -4\pi \rho} \quad \text{same as electrostatics!}$$

$$\Rightarrow \phi(\vec{r}, t) = \int d^3 r' \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|}$$

no matter what motion the source $\rho(\vec{r}, t)$ has!
 ϕ is given by the instantaneous Coulomb potential even though electromagnetic fields have a finite velocity of propagation c !

Ampere's Law becomes:

$$-\nabla^2 A + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c} \vec{j} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t})$$

$$\Rightarrow \nabla^2 A = \frac{4\pi}{c} \vec{j} - \frac{1}{c} \vec{\nabla}(\frac{\partial \phi}{\partial t})$$

$$\text{where } \vec{\nabla}(\frac{\partial \phi}{\partial t}) = \vec{\nabla} \left[\int d^3 r' \frac{\partial \rho}{\partial t} \frac{1}{|\vec{r} - \vec{r}'|} \right]$$

$$= - \vec{\nabla} \left[\int d^3 r' \frac{\vec{\nabla}' \cdot \vec{j}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} \right] \quad \text{by continuity eqn.}$$

To see the meaning of this term, recall - any vector function $\vec{j}$ can be written as the sum of a curlfree and a divergenceless part

$$\vec{j} = \vec{j}_{||} + \vec{j}_{\perp}$$

$$\text{where } \vec{\nabla} \times \vec{j}_{||} = 0$$

curlfree

$$\vec{\nabla} \cdot \vec{j}_{\perp} = 0$$

divergenceless

where

$$\vec{j}_{||}(\vec{r}) = -\frac{1}{4\pi} \vec{\nabla} \int d^3 r' \frac{\vec{\nabla}' \cdot \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

longitudinal part

$$\vec{j}_{\perp}(\vec{r}) = \frac{1}{4\pi} \vec{\nabla} \times \int d^3 r' \frac{\vec{\nabla}' \times \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

transverse part

$$= \frac{1}{4\pi} \vec{\nabla} \times \int d^3 r' \frac{\vec{\nabla}' \times \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\text{So } \vec{\nabla}(\frac{\partial \phi}{\partial t}) = 4\pi \vec{j}_{||}, \text{ and}$$

$$\nabla^2 A = \frac{4\pi}{c} \vec{j} - \frac{4\pi}{c} \vec{j}_{||} = \frac{4\pi}{c} \vec{j}_{\perp}$$