

Transverse + Longitudinal Parts of vector functions

To prove the preceding claim, $\vec{f} = \vec{f}_\parallel + \vec{f}_\perp$,
where $\vec{\nabla} \times \vec{f}_\parallel = 0$ and $\vec{\nabla} \cdot \vec{f}_\perp = 0$, we first proceed
to prove Helmholtz theorem.

Helmholtz Theorem: For a vector function $\vec{f}(\vec{r})$ if one
knows the divergence and curl of $\vec{f}$ then one
can ~~uniquely~~ uniquely determine $\vec{f}$ itself.

That is, if

$$\vec{\nabla} \cdot \vec{f} = 4\pi D(\vec{r}) \quad \text{where } D(\vec{r}) \text{ is a known scalar function}$$

$$\vec{\nabla} \times \vec{f} = 4\pi \vec{C}(\vec{r}) \quad \text{where } \vec{C}(\vec{r}) \text{ is a known vector function}$$

~~Then one can solve for~~

And if well defined boundary conditions on $\vec{f}$
are known (here we will assume $\vec{f}(\vec{r}) \rightarrow 0$ as $\vec{r} \rightarrow \infty$),
then there is a unique solution for $\vec{f}(\vec{r})$.

We prove this by construction!

Assume a solution of the form

$$\vec{f} = -\vec{\nabla}\phi + \vec{\nabla} \times \vec{W} \quad \text{where } \phi \text{ is a scalar and } \vec{W} \text{ a vector}$$

Now we show that we can find such a solution

First consider

$$\vec{\nabla} \cdot \vec{f} = -\nabla^2 \phi + \vec{\nabla} \cdot (\vec{\nabla} \times \vec{W}) = -\nabla^2 \phi + 0 = 4\pi D(\vec{r})$$

So $-\nabla^2 \phi = 4\pi D(\vec{r})$ This is just Poisson's eqn we saw in electrostatics

Solution when $\phi(\vec{r}) \rightarrow 0$ as $r \rightarrow \infty$ is given by

$$\phi(\vec{r}) = \int d^3r' \frac{D(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

Coulomb-like
integral solution

Now consider

$$\begin{aligned} \vec{\nabla} \times \vec{f} &= -\vec{\nabla} \times \vec{\nabla} \phi + \vec{\nabla} \times (\vec{\nabla} \times \vec{W}) = 0 - \nabla^2 \vec{W} + \vec{\nabla} (\vec{\nabla} \cdot \vec{W}) \\ &= 4\pi \vec{C}(\vec{r}) \end{aligned}$$

Choose a gauge in which $\vec{\nabla} \cdot \vec{W} = 0$ (just like Coulomb gauge in magnetostatics)

$$\text{Then } -\nabla^2 \vec{W} = 4\pi \vec{C}(\vec{r})$$

$$\vec{W}(\vec{r}) = \int d^3r' \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

just like solution
for vector pot $\vec{A}$
in magnetostatics

So we have constructed a solution

$$\vec{f}(\vec{r}) = -\vec{\nabla} \phi + \vec{\nabla} \times \vec{W}$$

$$= -\vec{\nabla} \int d^3r' \frac{D(\vec{r}')}{|\vec{r} - \vec{r}'|} + \vec{\nabla} \times \int d^3r' \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\text{where } \vec{\nabla} \cdot \vec{f} = 4\pi D \text{ and } \vec{\nabla} \times \vec{f} = 4\pi \vec{C}$$

Note: For above solution to be well defined, the integrals must converge. They will converge if the "sources" $D(\vec{r})$ and $\vec{C}(\vec{r})$ are sufficiently "localized" in space, i.e. $D(\vec{r}) \rightarrow 0$, $\vec{C}(\vec{r}) \rightarrow 0$ sufficiently fast as $\vec{r} \rightarrow \infty$.

Now we show that the above solution is unique.

Suppose there was another solution $\vec{g}$ such that

$$\vec{\nabla} \cdot \vec{g} = 4\pi D \quad \text{and} \quad \vec{\nabla} \times \vec{g} = 4\pi \vec{C}$$

Consider $\vec{h} \equiv \vec{f} - \vec{g}$ then

$$\vec{\nabla} \cdot \vec{h} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{h} = 0$$

Can show that only such $\vec{h}$ that also has $\vec{h}(\vec{r}) \rightarrow 0$ as $\vec{r} \rightarrow \infty$ is $\vec{h} \equiv 0$, so $\vec{g} = \vec{f}$ and solution is unique.

As a consequence of Helmholtz Theorem we have also shown the following

- ① Any vector function $\vec{f}$ can be written in terms of a scalar and vector potential

$$\vec{f} = -\vec{\nabla} \varphi + \vec{\nabla} \times \vec{W}$$

or equivalently

② Any vector function $\vec{F}$ can be written in terms of a curl free and a divergenceless part

$$\vec{F} = \vec{F}_{||} + \vec{F}_{\perp} \quad \text{where} \quad \vec{\nabla} \times \vec{F}_{||} = 0 \quad \text{curl free}$$

$$\vec{\nabla} \cdot \vec{F}_{\perp} = 0 \quad \text{divergenceless}$$

$$\text{where} \begin{cases} \vec{F}_{||}(\vec{r}) = -\vec{\nabla} \phi(\vec{r}) = -\vec{\nabla} \int \frac{d^3 r'}{4\pi} \frac{[\vec{\nabla}' \cdot \vec{F}(\vec{r}')] }{|\vec{r} - \vec{r}'|} \\ \vec{F}_{\perp}(\vec{r}) = \vec{\nabla} \times \vec{W}(\vec{r}) = \vec{\nabla} \times \int \frac{d^3 r'}{4\pi} \frac{[\vec{\nabla}' \times \vec{F}(\vec{r}')] }{|\vec{r} - \vec{r}'|} \end{cases}$$

where in above we used $\phi(\vec{r}') = \frac{1}{4\pi} \vec{\nabla}' \cdot \vec{F}(\vec{r}')$

$$\vec{C}(\vec{r}') = \frac{1}{4\pi} \vec{\nabla}' \times \vec{F}(\vec{r}')$$

~~where~~ $\vec{F}_{||}$ is called the longitudinal part of $\vec{F}$

$\vec{F}_{\perp}$ is called the transverse part of $\vec{F}$

to understand the reason for these names, we need to consider the Fourier transforming

Above can be generalized to situations where $\vec{F}$ satisfies other boundary conditions, say has a specified value on a given boundary surface. One first replaces $\frac{1}{|\vec{r} - \vec{r}'|}$ by the appropriate Green's function — see more to come!

Discussion regarding Fourier transforms

$$\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \vec{f}(\vec{k}) \quad \text{Fourier transf}$$

$$\vec{f}(\vec{k}) = \int_{-\infty}^{\infty} d^3r e^{-i\vec{k}\cdot\vec{r}} \vec{f}(\vec{r}) \quad \text{inverse transf}$$

Some special cases well worth remembering

① Transform of Dirac function

$$\delta_{\vec{r}_0}(\vec{k}) \equiv \int d^3r e^{-i\vec{k}\cdot\vec{r}} \delta(\vec{r}-\vec{r}_0) = e^{-i\vec{k}\cdot\vec{r}_0}$$

$$\Rightarrow \delta(\vec{r}-\vec{r}_0) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \delta_{\vec{r}_0}(\vec{k})$$

$$\boxed{\delta(\vec{r}-\vec{r}_0) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}_0(\vec{r}-\vec{r}_0)}$$

or letting $\vec{r} \leftrightarrow \vec{k}$ in the above

$$\delta(\vec{k}-\vec{k}_0) = \int \frac{d^3r}{(2\pi)^3} e^{i\vec{r}\cdot(\vec{k}-\vec{k}_0)}$$

② Transform of Coulomb potential $\frac{1}{|\vec{r}-\vec{r}'|}$

We know

$$\nabla^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = -4\pi \delta(\vec{r}-\vec{r}')$$

Suppose $f(\vec{k}) \equiv \int d^3r e^{-i\vec{k}\cdot\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|}$ is the

Fourier transf of $\frac{1}{|\vec{r}-\vec{r}'|}$

Substitute $\left\{ \begin{aligned} \frac{1}{|\vec{r}-\vec{r}'|} &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} f(\vec{k}) \\ \delta(\vec{r}-\vec{r}') &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \end{aligned} \right.$

into above Poisson equation

$$\nabla^2 \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} f(\vec{k})$$

operates only on $\vec{r}$
so move inside integral

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla} \cdot (\vec{\nabla} e^{i\vec{k}\cdot\vec{r}})$$

$$\textcircled{1} \quad \vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^3 \hat{x}_i \frac{\partial}{\partial x_i} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^3 \hat{x}_i i k_i e^{i\vec{k}\cdot\vec{r}} = i\vec{k} e^{i\vec{k}\cdot\vec{r}} \quad \text{where } \hat{x}_1, \hat{x}_2, \hat{x}_3 = \hat{x}, \hat{y}, \hat{z}$$

$$\textcircled{2} \quad \vec{\nabla} \cdot (i\vec{k} e^{i\vec{k}\cdot\vec{r}}) = (i\vec{k}) \cdot (i\vec{k}) e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

$$\text{so } \nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

Poisson equation then gives

$$\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} (-k^2) f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} e^{-i\vec{k}\cdot\vec{r}'}$$

$$\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} [-k^2 f(\vec{k})] = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} [-4\pi e^{-i\vec{k}\cdot\vec{r}'}]$$

As is true for Fourier series, so it is true for Fourier transforms: If two functions are equal, then their Fourier transforms are equal.

$$\Rightarrow -k^2 f(\vec{k}) = -4\pi e^{-i\vec{k} \cdot \vec{r}'}$$

$$f(\vec{k}) = \frac{4\pi}{k^2} e^{-i\vec{k} \cdot \vec{r}'}$$

$\Rightarrow$ is the Fourier transform of $\frac{1}{|\vec{r} - \vec{r}'|}$

Returning to Ampere's law we see that the term

$$\vec{\nabla} \left(\frac{\partial \phi}{\partial t} \right) = -\vec{\nabla} \int d^3r' \left[\frac{\vec{\nabla}' \cdot \vec{j}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} \right]$$

$$= 4\pi \vec{j}_{||}(\vec{r}, t)$$

So Ampere's law becomes

$$\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j} - \frac{4\pi}{c} \vec{j}_{||}$$

$$\boxed{\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}_{\perp}}$$

In Coulomb gauge, only the transverse part of $\vec{j}$ serves as a source for $\vec{A}$.

$\vec{A}$ describes the transverse modes, i.e. the EM radiation (recall in EM waves, the fields are always $\perp$ direction of propagation)

ϕ describes the longitudinal modes

Coulomb gauge is not Lorentz invariant - if $\vec{\nabla} \cdot \vec{A} = 0$ in one inertial reference frame, in general $\vec{\nabla} \cdot \vec{A} \neq 0$ in another.

In Coulomb gauge, if $\rho = 0$, then $\phi = 0$ and

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

Electrostatic

$$-\nabla^2 \phi = 4\pi\rho \quad \text{with} \quad \vec{E} = -\vec{\nabla}\phi \quad (\text{statics only})$$

physical meaning of the potential ϕ

work done to move a test charge δq from $\vec{r}_1$ to $\vec{r}_2$ in presence of an electric field $\vec{E}$ is

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{F}$$

where $\vec{F}$ is the force required to move the charge.

Since $\vec{E}$ exerts a force $\delta q \vec{E}$ on the charge,

$\vec{F}$ must counterbalance this electric force so we can move the charge quasi statically $\Rightarrow \vec{F} = -\delta q \vec{E}$

$$W_{12} = -\delta q \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{E} = \delta q \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{\nabla}\phi = \delta q [\phi(\vec{r}_2) - \phi(\vec{r}_1)]$$

$$\phi(\vec{r}_2) - \phi(\vec{r}_1) = \frac{W_{12}}{\delta q}$$

difference in potential between two points is the work per unit charge to move a test charge between the two points