Green's Theorem, Uniqueness, Green function - part II

We want to show that the boundary value problem we described is well posed - ie there is a unique solution. We start by deriving Greens Theorems

Consider $\int_V d^3r\, \vec{\nabla}\cdot\vec{A} = \oint_S da\, \hat{n}\cdot\vec{A}$ Gauss theorem

let $\vec{A} = \phi\vec{\nabla}\psi$ $\quad \phi, \psi$ any two scalar functions

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{A} = \phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$$

$$\phi\,\vec{\nabla}\psi\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$

$$\Rightarrow \int_V d^3r\left(\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\,\phi\frac{\partial\psi}{\partial n} \quad \Big\}\text{ Green's 1st identity}$$

let $\phi \leftrightarrow \psi$

$$\int_V d^3r\left(\psi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\,\psi\frac{\partial\phi}{\partial n}$$

subtract

$$\int_V d^3r\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S da\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) \quad \Big\}\text{ Green's 2nd identity}$$

Apply Greens 2nd identity with $\psi = \frac{1}{|\vec{r}-\vec{r}'|}$, $\vec{r}'$ is integration variable, ϕ is the scalar potential with $\nabla^2\phi = -4\pi\rho$. Use $\nabla^2\psi = \nabla'^2\psi = -4\pi\,\delta(\vec{r}-\vec{r}')$

$$\int_V d^3r'\left[\phi(r')\left[-4\pi\,\delta(r-r')\right] - \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\left(-4\pi\rho(\vec{r}')\right)\right]$$

$$= \oint_S da'\left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'}\right]$$

If $\vec{r}$ lies within the volume V, then

$$(*) \qquad \phi(\vec{r}) = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

Note: if $\vec{r}$ lies outside the volume V, then

$$0 = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint \frac{da'}{4\pi}\left[\underbrace{\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'}} - \underbrace{\phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)}\right]$$

potential from a surface charge density $\sigma = \frac{1}{4\pi}\frac{\partial\phi}{\partial n'}$

potential from a surface dipole layer of dipole strength density $\frac{\phi}{4\pi}$

From $(*)$, if $S \to \infty$ and $E \sim \frac{\partial\phi}{\partial n} \to 0$ faster than $\frac{1}{r}$, then the surface integral vanishes and we recover Coulombs law $\phi(\vec{r}) = \int d^3r' \rho(\vec{r}')/|\vec{r}-\vec{r}'|$

$(*)$ gives the generalization of Coulombs law to a system with a finite boundary

For a charge free volume V, ie $\rho(r) = 0$ in V, the potential everywhere is determined by the potential and its normal derivative on the surface.

But one cannot in general freely specify both ϕ and $\frac{\partial\phi}{\partial n}$ on the boundary surface since the resulting ϕ from $(*)$ would not in general obey Laplace's equation $\nabla^2\phi = 0$.

Specifying both ϕ and $\frac{\partial \phi}{\partial n}$ on surface is known as "Cauchy" boundary conditions — for Laplace's eqn, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

<u>Uniqueness</u>

If we have a system of charges in vol V, and <u>either</u> the potential ϕ, <u>or</u> its normal derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of V, then there is a unique solution to Poisson's equation inside V. Specifying ϕ is known as Dirichlet boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as Neumann boundary conditions.

<u>proof</u>: Suppose we had two solutions ϕ_1 and ϕ_2, <u>both</u> with $-\nabla^2\phi = 4\pi\rho$ inside V, and obeying specified b.c. on surface of V.

Define $u = \phi_2 - \phi_1 \Rightarrow \nabla^2 u = 0$ inside V

and $u = 0$ on surface S - for Dirichlet b.c.

or $\frac{\partial u}{\partial n} = 0$ on surface S - for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = u$

$$\int_V d^3r\,(\underbrace{u\nabla^2 u}_{=0} + \vec\nabla u\cdot\vec\nabla u) = \oint_S da\,\underbrace{u\frac{\partial u}{\partial n}}_{=0}$$

as $\nabla^2 u = 0$ as u <u>or</u> $\frac{\partial u}{\partial n} = 0$

$$\Rightarrow \int_V d^3r \, |\vec{\nabla} U|^2 = 0 \qquad \Rightarrow \vec{\nabla} U = 0$$
$$\Rightarrow U = \text{const}$$

For Dirichlet b.c., $U = 0$ on surface S, so const $= 0$ and $\phi_1 = \phi_2$. Solution is unique

For Neumann b.c., ϕ_1 and ϕ_2 differ only by an arbitrary constant. Since $\vec{E} = -\vec{\nabla}\phi$, the electric fields $\vec{E}_1 = -\vec{\nabla}\phi_1$ and $E_2 = -\vec{\nabla}\phi_2$ are the same.

If boundary ~~~~ surface S consists of several disjoint pieces, then solution is unique if specify ϕ on some pieces and $\frac{\partial \phi}{\partial n}$ on other pieces.

Solution of Poisson's equation with both ϕ and $\frac{\partial \phi}{\partial n}$ specified on the same surface S (Cauchy b.c.) does not in general exist, since specifying either ϕ or $\frac{\partial \phi}{\partial n}$ alone is enough to give a unique solution.

Green's function - part II

Greens 2nd identity

$$\int_V d^3r' \left(\phi \nabla'^2 \psi - \psi \nabla'^2 \phi \right) = \oint_S da' \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right)$$

Apply above with $\begin{cases} \phi(\vec{r}') \text{ electrostatic potential with } \nabla'^2\phi = -4\pi\rho(\vec{r}') \\ \psi(\vec{r}') = G(\vec{r},\vec{r}') \text{ the Green function satisfying} \end{cases}$

$$\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$$

we saw one solution of above is $G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$
but a more general solution is

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}')$$

where $\nabla'^2 F(\vec{r},\vec{r}') = 0$, for $\vec{r}'$ in volume V
we will choose $F(\vec{r},\vec{r}')$ to simplify solution of ϕ

$$\Rightarrow \int_V d^3r' \left(\phi(\vec{r}') \nabla'^2 G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}') \nabla'^2 \phi(\vec{r}') \right)$$

$$= \int_V d^3r' \left(\phi(r') \left[-4\pi\delta(\vec{r}-\vec{r}') \right] - G(\vec{r},\vec{r}') \left[-4\pi\rho(\vec{r}') \right] \right)$$

$$= -4\pi\phi(\vec{r}) + 4\pi \int_V d^3r' \, G(\vec{r},\vec{r}')\rho(\vec{r}')$$

$$= \oint_S da' \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

$$\phi(\vec{r}) = \int_V d^3r'\, G(\bar{r},\bar{r}')\, \rho(\bar{r}') + \oint_S \frac{da'}{4\pi} \left(G(\bar{r},\bar{r}') \frac{\partial \phi(\bar{r}')}{\partial n'} - \phi(r') \frac{\partial G(\bar{r},\bar{r}')}{\partial n'} \right) \checkmark$$

Consider Dirichlet boundary problem. If we can choose $F(\bar{r},\bar{r}')$ such that $G_D(\vec{r},\vec{r}') = 0$ for $\vec{r}'$ on the boundary surface S, then above simplifies to

$$\boxed{\phi(\vec{r}) = \int_V d^3r'\, G_D(\bar{r},\vec{r}')\, \rho(\bar{r}') - \oint_S \frac{da'}{4\pi}\, \phi(r') \frac{\partial G_D(\bar{r},\vec{r}')}{\partial n'}}$$

Since $\rho(r)$ is specified in V, and $\phi(r)$ is specified on S, above then gives desired solution for $\phi(r)$ inside volume V.

Finding G_D is therefore equivalent to finding an $F(r,r')$ such that $\nabla'^2 F(\bar{r},\bar{r}') \stackrel{?}{=} 0$ for $\vec{r}'$ in V (solves Laplace eqn) and

$F(\vec{r},\vec{r}') = \dfrac{-1}{|\bar{r}-\bar{r}'|}$ for $\vec{r}'$ on boundary surface S'

Always exists unique solution for F

Next consider Neumann boundary problem.

One might think to find $F(\vec{r},\vec{r}\,')$ such that $\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} = 0$ on boundary surface. But this is not possible.

Consider $\int_V \nabla'^2 G(\vec{r},\vec{r}\,')\, d^3r' = \int \vec{\nabla}' \cdot \vec{\nabla}' G(\vec{r},\vec{r}\,')\, d^3r'$

$$= \oint_S \vec{\nabla}' G(\vec{r},\vec{r}\,') \cdot \hat{n}\, da'$$

$$= \oint_S \frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'}\, da' = -4\pi \qquad \text{since } \nabla'^2 G = -4\pi\delta(r-r')$$

So we can't have $\frac{\partial G}{\partial n'} = 0$ for $\vec{r}\,'$ on S

Simplest choice is then $\frac{\partial G_N(r,r')}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{r}\,'$ on S

area of surface

Then

$$\phi(\vec{r}) = \int_V d^3r'\, G_N(\vec{r},\vec{r}\,')\, \rho(\vec{r}\,') + \oint \frac{da'}{4\pi}\, G(\vec{r},\vec{r}\,')\, \frac{\partial\phi(\vec{r}\,')}{\partial n'}$$

$$+ \oint \frac{da'}{4\pi}\, \phi(r') \left(\frac{-4\pi}{S}\right)$$

$$\left[\phi(\vec{r}\,') = \int_V d^3r'\, G_N(\vec{r},\vec{r}\,')\, \rho(\vec{r}\,') + \oint \frac{da'}{4\pi}\, G_N(\vec{r},\vec{r}\,')\, \frac{\partial\phi(\vec{r}\,')}{\partial n'}\right]$$

$$+ \langle\phi\rangle_S$$

constant = average value of ϕ on surface S.

Since $\rho(r)$ is specified in V and $\frac{\partial\phi}{\partial n}$ is specified on S

above gives solution $\phi(r)$ in V within additive constant $\langle\phi\rangle_S$

Since $\vec{E} = -\vec{\nabla}\phi$, the const $\langle\phi\rangle_S$ is of no consequence

Finding $G_N(\vec{r},\vec{r}')$ is therefore equivalent to finding an $F(\vec{r},\vec{r}')$ such that

$$\nabla'^2 F(\vec{r},\vec{r}') = 0 \quad \text{for } \vec{r}' \text{ in } V$$

and $$\frac{\partial F(\vec{r},\vec{r}')}{\partial n'} = -\frac{4\pi}{S'} \quad \text{for } \vec{r}' \text{ on surface } S'$$

always exists a unique solution (within additive constant)

While G_D and G_N always exist in principle, they depend in detail on the shape of the surface S and are difficult to find except for simple geometries

In preceeding we defined G by $\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$

But our earlier interpretation of $G(\vec{r},\vec{r}')$ was that it was potential at $\vec{r}$ due to point source at $\vec{r}'$, ie $\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$. Note, for general surface S', $G(\vec{r},\vec{r}')$ is not in general a function of $|\vec{r}-\vec{r}'|$ but depends on $\vec{r}$ and $\vec{r}'$ separately. But the equivalence of the two definitions of G above is obtained by noting that one can prove the symmetry property

$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$$

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c. (see Jackson, end section 1.10)