

Separation of Variables

If the system has a rectangular boundary, and contains no charge, we can look for solutions to $\nabla^2 \phi = 0$ of the form

$$\phi(r) = X(x) Y(y) Z(z)$$

product of three functions
each of one variable only

$$\nabla^2 \phi = 0 \Rightarrow \frac{1}{\phi} \nabla^2 \phi = 0$$

$$\Rightarrow \frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z}{dz^2} = 0$$

The only way this can happen for all values of x, y, z is if each of the three terms is a constant, call them a^2, b^2, c

$$\frac{1}{X} \frac{d^2 X}{dx^2} = a^2 \Rightarrow X(x) = A_1 e^{-ax} + A_2 e^{ax}$$

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = b^2 \Rightarrow Y(y) = B_1 e^{-by} + B_2 e^{by}$$

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = c^2 \Rightarrow Z(z) = C_1 e^{-cz} + C_2 e^{cz}$$

$$\text{with } a^2 + b^2 + c^2 = 0$$

$\Rightarrow$ at least one of the a^2, b^2, c^2 is < 0

$\Rightarrow$ at least one of the a, b, c is imaginary.

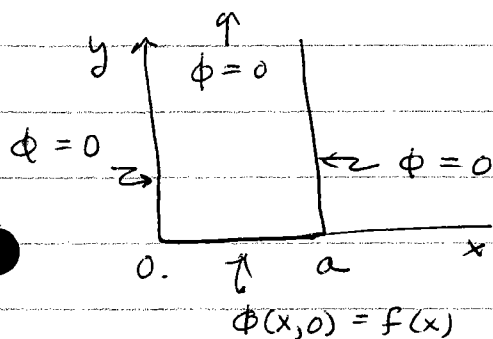
Above is one particular solution. But there are many solutions, each with different a, b, c , but all obeying the constraint $a^2 + b^2 + c^2 = 0$. The General solution is a superposition of these

$$\phi(x, y, z) = \sum_i (A_{1i} e^{-a_i x} + A_{2i} e^{a_i x}) (B_{1i} e^{-b_i y} + B_{2i} e^{b_i y}) (C_{1i} e^{-c_i z} + C_{2i} e^{c_i z})$$

Example

$$a_i^2 + b_i^2 + c_i^2 = 0$$

Consider a channel shaped as below - infinite along z



$$\phi(0, y) = 0$$

$$\phi(a, y) = 0$$

$$\phi(x, y) = 0 \text{ as } y \rightarrow \infty$$

$$\phi(x, 0) = f(x) \text{ specified function}$$

solution is independent of $z \Rightarrow$

$$\phi(x, y) = \sum_i (A_{1i} e^{-a_i x} + A_{2i} e^{a_i x}) (B_{1i} e^{-b_i y} + B_{2i} e^{b_i y})$$

$$a_i^2 + b_i^2 = 0$$

we will see that the correct thing is to choose a magnitude

$$\text{let } a_i = i\alpha_i$$

$$b_i = \alpha_i$$

$$\phi(x, y) = \sum_i (A_i \cos \alpha_i x + B_i \sin \alpha_i x) (C_i e^{-\alpha_i y} + D_i e^{\alpha_i y})$$

$$\text{where } A_i = A_{1i} + A_{2i}$$

$$C_i = B_{1i}$$

$$B_i = i(A_{1i} - A_{2i})$$

$$D_i = B_{2i}$$

Now $\phi(x, y) \rightarrow 0$ as $y \rightarrow \infty$ for all $x \Rightarrow \boxed{D_2 = 0}$

$$\Rightarrow \phi(x, y) = \sum_i \left[A_i' \cos \alpha_i x + B_i' \sin \alpha_i x \right] e^{-\alpha_i y}$$

$$\text{where } A_i' = A_i C_i, \quad B_i' = B_i C_i$$

$$\phi(0, y) = 0 \Rightarrow \sum_i A_i' e^{-\alpha_i y} = 0 \text{ all } y \Rightarrow \boxed{A_i' = 0}$$

$$\Rightarrow \phi(x, y) = \sum_i B_i' \sin(\alpha_i x) e^{-\alpha_i y}$$

$$\phi(a, y) = 0 \Rightarrow \sum_i B_i' \sin(\alpha_i a) e^{-\alpha_i y} = 0 \text{ all } y$$

$$\Rightarrow \sin(\alpha_i a) = 0 \text{ or } \alpha_i a = n\pi$$

$$\alpha_i = \frac{n\pi}{a} \text{ integer } n \geq 1$$

$$\Rightarrow \boxed{\phi(x, y) = \sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}}$$

Finally

$$\phi(x, 0) = f(x) \Rightarrow \sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) = f(x)$$

This is just the Fourier series for $f(x)$!

$$\boxed{B_n' = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx}$$

We have thus determined all unknown coefficients and found the solution

See Jackson 2-8 if
Fourier series needs review

Recall orthogonality : $\frac{2}{a} \int_0^a dx \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

For $f(x) = \phi_0$ a constant,

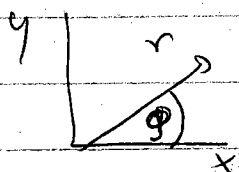
$$B_n' = \frac{2}{a} \phi_0 \int_0^a dx \sin\left(\frac{n\pi x}{a}\right) = \frac{2\phi_0}{a} \left[-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right) \right]_0^a$$

$$= \frac{2\phi_0}{n\pi} (1 - \cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \frac{4\phi_0}{n\pi} & n \text{ odd} \end{cases}$$

Polar Coordinates

- still translationally invariant along z - so two dimensional

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \varphi^2} = 0$$



assume $\phi(r, \varphi) = R(r) \Phi(\varphi)$

$$\frac{r^2 \nabla^2 \phi}{\phi} = \frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = 0$$

each term must be a constant

$$\Rightarrow \frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = \nu^2, \quad \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = -\nu^2$$

$$\left. \begin{aligned} \text{Solutions are } R(r) &= ar^\nu + br^{-\nu} \\ \Phi(\varphi) &= A \cos(\nu\varphi) + B \sin(\nu\varphi) \end{aligned} \right\} \nu \neq 0$$

$$\left. \begin{aligned} R(r) &= a_0 + b_0 \ln r \\ \Phi(\varphi) &= A_0 + B_0 \varphi \end{aligned} \right\} \nu = 0$$

~~$$\phi(r, \varphi) = (a_0 + b_0 \ln r) (A_0 + B_0 \varphi) + \sum_{n=1}^{\infty} \left[r^n (A_n \cos(n\varphi) + B_n \sin(n\varphi)) + r^{-n} (C_n \cos(n\varphi) + D_n \sin(n\varphi)) \right]$$~~

If φ can take its entire range from 0 to 2π (such as problem in which ϕ is specified on the surface of a cylinder) then periodicity in $\varphi \rightarrow \varphi + 2\pi$ requires $B_0 = 0$ and $\nu = \text{integer } n$

$$\phi = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[r^n (A_n \cos(n\varphi) + B_n \sin(n\varphi)) + r^{-n} (C_n \cos(n\varphi) + D_n \sin(n\varphi)) \right]$$

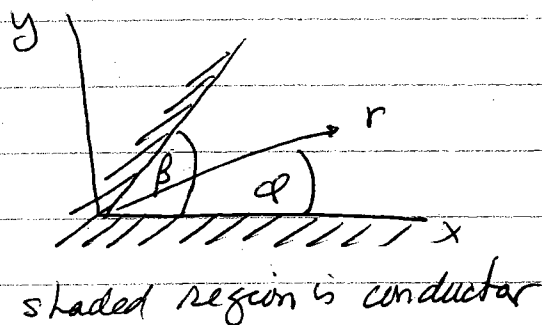
or reparameterizing

$$\phi(r, \varphi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[a_n r^n \sin(n\varphi + \alpha_n) + b_n r^{-n} \sin(n\varphi + \beta_n) \right]$$

If the region where there is no charge includes $r=0$, then all $b_n = 0$ since ϕ should not diverge at the origin.

If $r=0$ is excluded from the region, then the b_n need not be zero. The case $b_0 \neq 0$ corresponds to a line charge λ along the z axis.

Consider the case where φ has a restricted range, for example a wedge shaped opening of angle β



$$0 \leq \varphi \leq \beta$$

ϕ is constant in conductor
 $\Rightarrow$ boundary conditions

$$\begin{cases} \phi(r, 0) = \phi_0 \\ \phi(r, \beta) = \phi_0 \end{cases}$$

The general solution is the linear combination

$$\phi(r, \varphi) = (a_0 + b_0 \ln r)(A_0 + B_0 \varphi)$$

$$+ \sum_{\nu > 0} (a_\nu r^\nu + b_\nu r^{-\nu})(A_\nu \cos(\nu \varphi) + B_\nu \sin(\nu \varphi))$$

- ① The condition $\phi(r, 0) = \phi_0$ a constant indep of r then requires

$$b_0 = 0, A_\nu = 0 \text{ all } \nu$$

So

$$\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{\nu > 0} (a_\nu r^\nu + b_\nu r^{-\nu}) B_\nu \sin(\nu \varphi)$$

- ② Since ϕ should be continuous as one approaches the conducting surface, and $\phi = \phi_0$ is a finite constant on the conducting surface, then ϕ cannot diverge as one approaches the origin $r=0$ along any fixed angle φ . This requires

$$b_\nu = 0 \text{ all } \nu$$

So

$$\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{\nu > 0} a_\nu r^\nu \sin(\nu \varphi)$$

- ③ The condition $\phi(r, \beta) = \phi_0$ a constant independent of r then requires

$$\sin(\nu \beta) = 0 \Rightarrow \nu = \frac{n\pi}{\beta}, \quad n \text{ integer } \geq 1$$

So

$$\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi \varphi}{\beta}\right)$$

- ④ as ϕ must approach the constant ϕ_0 as $r \rightarrow 0$ along any fixed angle φ , we therefore must have

$$B_0 = 0, \quad a_0 A_0 = \phi_0$$

So finally we have

$$\phi(r, \varphi) = \phi_0 + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi\varphi}{\beta}\right)$$

We still have all the unknown a_n ! These depend on how $\phi(r, \varphi)$ behaves as $r \rightarrow \infty$ (we can't make the choice here that $\phi \rightarrow 0$ as $r \rightarrow \infty$) - this is additional information that must be specified to find the complete solution.

Nevertheless we can still get very interesting information near the origin at small r . In this case, the leading term in the above series expansion for ϕ is the $n=1$ term, as it vanishes most slowly as $r \rightarrow 0$.

$$\phi(r, \varphi) \simeq \phi_0 + a_1 r^{\frac{\pi}{\beta}} \sin\left(\frac{\pi\varphi}{\beta}\right)$$

The electric field is

$$E_r(r, \varphi) = -\frac{\partial \phi}{\partial r} = -\frac{\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \sin\left(\frac{\pi\varphi}{\beta}\right)$$

$$E_\varphi(r, \varphi) = -\frac{1}{r} \frac{\partial \phi}{\partial \varphi} = -\frac{\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \cos\left(\frac{\pi\varphi}{\beta}\right)$$

$$\Rightarrow \boxed{E \sim r^{\frac{\pi}{\beta}-1}}$$

Induced surface charge given by $4\pi\sigma = \vec{E} \cdot \hat{n}$

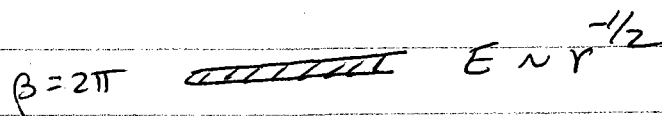
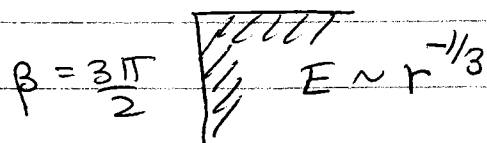
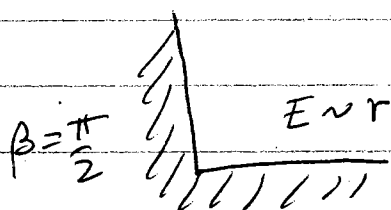
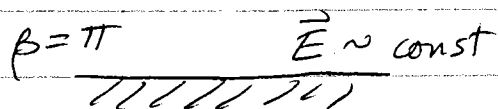
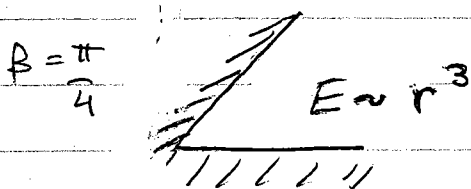
for surface at $\varphi=0$, $\hat{m} = \hat{\varphi}$
 for surface at $\varphi=\beta$, $\hat{m} = -\hat{\varphi}$

$$\sigma(r, \varphi=0) = \frac{E\varphi(r,0)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta}-1}$$

$$\sigma(r, \varphi=\beta) = -\frac{E\varphi(r,\beta)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta}-1}$$

For $\frac{\pi}{\beta} > 1$, i.e. $\beta < \pi$, $\vec{E}$ and σ vanish as approach the origin.

For $\frac{\pi}{\beta} < 1$, i.e. $\beta > \pi$, $\vec{E}$ and σ diverge as approach the origin.



E diverges at an "external" corner

E vanishes at an "internal" corner

Remember, the above examples all had translational symmetry along z , so the "corners" above are really infinitely long straight "edges".